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GAME THEORETIC MODELS OF HIGH ORDER NETWORK INTERACTIONS

BY

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To my grandfather Franco, who is not here to see this day, but who envisioned it many years ago and made it possible.

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To all the people who have shared this winding journey with me, whether they are my center of gravity or they passed by like comets, each of you has contributed to this achievement and to making me the person I am today. Thank you.

DECLARATION

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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ABSTRACT

In this thesis we present the notion of separable game, a game theoretic representation based on forward directed hypergraphs (FDH-graphs) that incorporates both features of locality and separability characterizing real world network systems. We perform an in depth analysis of this representation, which refines the popular model of graphical games.

First, we address the problem of giving an expressive representation of games without giving up on conciseness. Our main contributions in this sense are the following. We show that there exists a minimal FDH-graph with respect to which a game is separable, providing a minimal complexity description for the game that models joint interactions among groups of players. Moreover, we propose a geometric characterization of separability that results in checkable conditions and algorithms to identify the minimal FDH-graph of a game. We discuss both modelling and computational complexity issues related to the separable representation of games, proposing separability as a way to capture essential interactions in games and the related concept of strict-separability as a way to obtain more compact representations.

Then, we address the problem of measuring how far a given game is from being separable with respect to a given FDH-graph and the related problem of computing the best approximation of a game with some desired separability property. Both these problems are solved by means of games projections, which provide explicit formulas for both tasks.

Finally, we focus on structural properties of separable games, i.e., we investigate how the hypergraphical structure of games reflects on their properties. For the class of potential games we prove a symmetry property of the minimal FDH-graph and we derive an exact correspondence between the separability property of a game and the decomposition of its potential function in terms of local functions. These results strengthen the ones recently proved for graphical potential games [14]. We also study the interplay between separability and the decomposition of games in their harmonic and potential components [31], characterizing the separability properties of both such components. For generic games we show that the structure of correlated equilibria is shaped by that of the game's FDH-graph, by proving that, up to a notion of equivalence, correlated equilibria of separable games can be factorized based on the grouping of players described by hyperlinks. We then discuss the computational and descriptive advantages that result from this correspondence.

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1 INTRODUCTION

1.1 MOTIVATION

In this thesis we undertake a fundamental analysis of separable games, a representation that generalizes other existing graphical models for game theory that have recently emerged as a unified framework for modelling interactions in many social and economic settings.

The main feature of these settings is that interactions are based on a principle of *locality*, as the decisions of individuals are affected by the actions of their friends, colleagues, peers, or competitors. Another common feature is that of *separability*, as the effects on an individual can often be described as the superimposition of various partial effects due to particular subsets of agents.

The most common way to formalize the concept of locality is through the notion of graphs. For this reason graph based representations of games have emerged as a framework to describe the rich variety of real world multi-agent systems that are ruled by small scale strategic interactions. The core structure of all such systems can be effectively captured by a graph, which encodes both the pattern of interactions among entities and their local nature. This has stimulated a fruitful investigation of network game representations, which has mainly focused on the two most popular models, namely graphical polymatrix [37] and graphical games [59]. The rich literature on both kinds of representations, which will be presented in the next section, amply demonstrates the positive contributions that these models gave to the progress of various research areas. These include, but are not limited to, the characterization of structural properties of important classes of games in terms of their network structure, the investigation of how key features of specific game models result from the topology of the underlying interaction network, and the derivation of computational procedures for game theory that take advantage of the graphical representation both to compactly represent data and to efficiently process it.

Classical network representations for games, however, have a strong limitation, which has been highlighted by a recent and increasingly rich literature on complex systems [3, 18, 20]: they lack the expressive power needed to represent separability of interactions. Graphs represent interactions among entities, corresponding to nodes, by means of links, which are inherently binary objects. Based on nodes and links, graphs allow to describe either a completely joint or a completely pairwise dependence of an entity on the other entities it is connected to. As a consequence, graphs fail to represent more complex forms of high-order interactions, which have been experimentally shown to characterize the behavior of social systems [67], biological systems [66, 46] and more generally of complex systems [19, 63].

Therefore, the need has emerged for new game representations that allow a more expressive representation of high-order interactions, going beyond and filling the gap between the pairwise and joint perspective. A natural mathematical object to support the creation of new models with the desired expressive power are hypergraphs, which incorporate both features of locality and separability thus allowing modelling of joint interactions among groups of entities by means of hyperlinks. Higher-order analysis of network systems is of growing importance in many research fields and has already produced fruitful results in the physics and complex systems communities. For example, competitive network models including higher-order interactions have provided a formal framework to explain the stable coexistence of many communities and different species in natural ecosystems exhibiting rich biodiversity [15, 48]; higher-order models of social contagion have been able to reproduce emerging phenomena observed in the process of opinion formation, diffusion of behaviors and epidemic spread in complex social systems that cannot be justified in terms of pairwise interactions [51, 40, 23]; dynamical systems on hypergraphs [32], such as systems of coupled oscillators with non-pairwise interactions [22], have advanced the understanding of collective behavior of real world systems, including brain networks and protein interaction networks.

Within the game theory community, hypergraphical games [80, 76] have been proposed as a game representation to describe strategic interactions among groups of players. However, this model has not yet been extensively explored and only a limited amount of literature is available on the topic. Hypergraphical games have been introduced in the context of computational game theory as an appealing representation of the input data for algorithmic procedures. In this setting the hypergraphical structure is not investigated as the focus is on the succinctness property of such description, which is not especially related to hypergraphs but is shared by many different game representations. To the best of our knowledge, just a few works have explored the properties of hypergraphical games from a modelling perspective, mainly focusing on high-order models of coordination or mis-coordination [81, 82], voter models and more general models of consensus and coalition formation [33, 6], evolutionary dynamics for public good provision in social systems with higher-order interactions [7].

Still, hypergraphs have a drawback. They are characterized by a symmetry property which makes them most suitable to represent high-order undirected interactions, i.e., situations where if an entity A is jointly interacting with B and C, then both B and C interact with A and with each other. This symmetry property is clearly not always satisfied in real world systems but hypergraphical models are unable to capture its absence, possibly failing to represent explicitly an important feature of the system.

This motivates our interest in separable games, the main subject of this dissertation, which offer a representation of games that can render high-order non necessarily symmetric interactions. Separable games are based on forward directed hypergraphs, which are the extension of hypergraphs from the undirected to the directed setting. The notion of separable games is not completely novel, as it is equivalent to that of graphical multi-hypermatrix games introduced in [77]. However, [77] considers graphical multi-hypermatrix games only from a computational perspective. In this thesis we undertake a different approach. Motivated by the interest of computational game theory in games endowed with hypergraphical structure and by the lack of an in-depth analysis of such models, we perform a systematic study of representational and structural properties of separable games and we explore the relation with other representations of games, showing how important results obtained in the graphical setting can be improved in the language of directed hypergraphs, thus leading the way for further future investigations.

1.2 BACKGROUND

The standard way of representing games is by their normal form. This representation consists in a table specifying the utility attained by each player in each configuration of the game. Despite being an intuitive representation that closely matches the theoretical definition of a game, it has two major shortcomings. First, it is non-compact as the size of the representation is exponential in the number of players. This feature undermines the development of efficient algorithms for solving game theoretic problems. Secondly, it doesn't allow to encode explicitly the strategic interactions among players. While for a generic game each player's utility is affected by the actions of every other player, this is not the case in many game theoretic models of real world systems, like social or economic networks. By ignoring this information we miss an important aspect of strategic interactions among players, which often shapes the game's properties. To overcome these limitations a number of alternative game representations have been proposed. Among the most relevant we cite multi-agent influence diagrams [60], game networks [62], action-graph games [57] and, most importantly, a series of models that we will collectively refer to as "network game representations". The variety of different models is due to the fact that each representation is most suitable for certain classes of games and it provides different insights on the represented game's properties.

Network game representations are based on some network structure encoding the strategic dependencies among players and have been by far the most impactful and fruitful models. These include graphical polymatrix games, graphical games, hypergraphical games, which will be briefly described in the rest of this section, and separable games, which are the subject of this work and will be introduced in Chapter 3.

1.2.1 *Polymatrix games*

Polymatrix games, first introduced in Russian in [55] and then in English in [50], are systems of independent pairwise interactions among players. They are the simplest extension of two-player games to the multi-player setting, obtained by linearly combining two-player games involving each couple of players. This construction gives rise to multi-player games in which each player is influenced separately by every other player. Formally, a polymatrix game u is characterized by the fact that the utility of each player $i \in \mathcal{V}$ can be expressed as the pointwise sum

$$u_i(x) = \sum_{j \in \mathcal{V} \setminus \{i\}} u_{ij}(x_i, x_j) \quad \forall x \in \mathcal{X} \tag{1}$$

of pairwise utility functions $u_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathbb{R}$. Such payoff decomposition drastically lightens the game description: by ruling out any higher order interactions, the description of the game is reduced to a collection of small, two player games. An even more compact description arises when incorporating explicitly the pattern of pairwise interactions into the game representation. In a *graphical polymatrix game* [37] (also referred to as separable network game [30]) on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (which is in general directed), the utility of each player $i \in \mathcal{V}$ is represented in the form

$$u_i(x) = \sum_{j \in \mathcal{N}_i} u_{ij}(x_i, x_j) \quad \forall x \in \mathcal{X}\,,$$

where a function $u_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathbb{R}$ is associated to each edge $(i, j) \in \mathcal{E}$. In particular, when $\mathcal{G}$ is undirected such games can be interpreted as follows. Players are identified with nodes of the graph and each pair of players $\{i, j\}$ connected by a link is involved in a two-player game having utility functions $u_{ij}(x_i, x_j)$ and $u_{ji}(x_j, x_i)$. Each player $i \in \mathcal{V}$ can choose an unique action $x_i \in \mathcal{A}_i$ to be used in all games they simultaneously participate in and they get a utility that is the linear aggregate of utilities from their outgoing links.

1.2.2 *Graphical games*

Graphical games, first introduced in [59], are defined by imposing more general restrictions on the way players' utilities depend on strategies. Precisely, a game u is said to be graphical on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if the utility of each player $i \in \mathcal{V}$ depends only on their own action and the actions of fellow players in their neighborhood in $\mathcal{G}$, i.e., if

$$u_i(x) = u_i(y) \tag{2}$$

for all game's configurations $x, y \in \mathcal{X}$ such that their restriction to the player's neighborhood coincide, namely, $x_{\mathcal{N}_i^\bullet} = y_{\mathcal{N}_i^\bullet}$. Clearly, any graphical polymatrix game can be represented as a graphical game on the same graph, but the latter representation loses track of the pairwise structure of each player's interactions with their neighbors. Viceversa, only graphical games that decompose pairwise on links can be represented as graphical polymatrix games. Indeed, the graphical representation of games captures locality of interactions according to a graph structure but allows for any kind of dependence for a player on their neighbors' actions, which include pairwise interactions as well as higher-order, joint interactions. As a consequence, graphical games are a more general representation of games than polymatrix. In fact, graphical games allow to represent any normal form game, as any game is trivially graphical on the complete graph on its player set $\mathcal{V}$. However, this representation is most interesting when it holds with a significantly sparser graph, which is often the case in applications.

1.2.3 *Hypergraphical games*

In the hypergraphical representation of games [80, 76] the utility of a player $i \in \mathcal{V}$ is expressed as the linear combination of payoff contributions associated to all hyperlinks of an undirected hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ that they belong to:

$$u_i(x) = \sum_{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}} u_i^{\mathcal{J}}(x_{\mathcal{J}}), \quad \forall x \in \mathcal{X}.$$

We can interpret each hyperlink $\mathcal{J}$ of $\mathcal{H}$ as a group of players that interact with each other in a local game, restricted only to players $i \in \mathcal{J}$ belonging to the hyperlink, where the payoff of each player is given by $u_i^{\mathcal{J}}$. This definition describes the fact that players may have separate interactions with different groups of other players: interactions within each group can be complex and are described by local games, while different groups affect a single player additively. It then follows that both graphical polymatrix and graphical games can be represented as hypergraphical games maintaining the same conciseness. Indeed, graphical polymatrix games on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be expressed as hypergraphical games on a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ where hyperlinks represent links of $\mathcal{G}$, i.e., $\mathcal{L} = \{\{i, j\} : (i, j) \in \mathcal{E}\}$, and local payoff contributions coincide with the corresponding pairwise utility functions. On the other hand, every graphical game on $\mathcal{G}$ can be expressed as a hypergraphical game on a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ where hyperlinks represent neighborhoods of $\mathcal{G}$, i.e., $\mathcal{L} = \{\mathcal{N}_i^\bullet, i \in \mathcal{V}\}$.

The hypergraphical game representation is characterized by an intrinsic symmetry, which follows from the fact that the underlying hypergraph is undirected. This translates to a symmetry of interactions, in that if a player i depends on two players j and k belonging to a common hyperlink, not only j and k depend on i, but they also interact with each other. A related representation is that of graphical multi-hypermatrix games [77], which relaxes the symmetry constraints by endowing each player with a possibly different undirected hypergraph describing the decomposition of their utility. This recent model has already proven its power in the computational setting of [77], but an analysis of its properties and modelling capabilities is still lacking. We tackle this point in this thesis by studying separable games, a notion that is equivalent to graphical multi-hypermatrix games aside from some minor technical differences (discussed in Remark 3.1.4).

1.3 LITERATURE REVIEW

Among the rich literature on network game models we can identify three major research areas of particular relevance for our discussion. The first research line investigates the representational power of the different models and focus on deriving general properties of classes of games based on their network structure. The second is an algorithmic line of research and focuses on exploiting the succinctness of games' representations to develop computationally efficient algorithms for solving games. The third line, instead, focuses on specific game models, often emerging from real world applications, and aims at obtaining novel insights and results based on the pattern of interactions among players. In this section, we tackle each of the above lines providing a brief summary of the related research and of the key results in the literature.

1.3.1 *Representational and structural properties of games*

By research on game representations we mean the literature that investigates how the structure of a game reflects on its properties. In general, structural properties are properties that can be proven to hold for all games that share a certain structure. In particular, in the context of network representations of games, the focus is on understanding which properties of games can be described in terms of the game's network structure. By establishing a relation between the network representation of a game and some of its relevant features, it is possible to obtain a representation as concise and informative for the features as the one for the game itself. This is beneficial both from an algorithmic perspective, as it opens the way to compute efficiently such features by exploiting graph algorithms, and from a more descriptive perspective, as it offers an interpretation of the game's features in terms of the pattern of its strategic interactions.

To clarify these considerations and without any aim to be exhaustive, we present three works that have produced results in this direction.

Polymatrix games made it possible to extend to the context of multi-player games major results that were known to hold for two-player games, but that don't hold for general, unstructured, multi-player games. A remarkable example is provided by [30] in the setting of zero-sum games. Two-players zero-sum games have been deeply and extensively studied since the early days of game theory, and they are still today. Zero sum games are defined as games u (see (1)) such that for each configuration $x \in \mathcal{X}$ of the game the sum of all utilities of players $i \in \mathcal{V}$ is vanishing

$$\sum_{i \in \mathcal{V}} u_i(x) = 0.$$

They represent a simple but effective model of strategic competition among players with opposite interests, in settings which can be modelled as closed payoff systems, so that a player's gain corresponds to an equivalent loss for the other players and viceversa. This intuitive interpretation makes zero-sum games very interesting from the point of view of applications. From a mathematical perspective, the zero-sum property turns out to be a crucial feature, the key element to obtain the strong and significant results which characterize two-player zero-sum games. The most relevant of such results is the so called "Minimax Theorem", due to Von Neumann [75]. Von Neumann's result not only proved the existence of mixed equilibria configuration, anticipating the work of Nash [74], but it also provides a characterization of equilibria in terms of minimax strategies. Such characterization, which is missing for general non zero-sum games, reduces the problem of finding Nash equilibria in two-player zero-sum games to that of solving a linear program, which can be done efficiently. This remarkable result is in sharp contrast with more recent results [35, 38], showing that the computation of Nash equilibria is a hard problem in general. The rich theory of zero-sum games was confined to the two-player setting. Indeed, even if the zero-sum property admits a straightforward extension, the minimax theorem fails in the general multi-player setting. In [37] the authors extend the minimax theorem to a subclass of graphical polymatrix games where a zero-sum two-player game involving the endpoints players is associated to each link. This construction, based on local two-player zero-sum games, results in a zero-sum polymatrix game but doesn't cover the whole family of zero-sum polymatrix games. The extension process is completed in [30], where the minimax theorem is proved for zero-sum graphical polymatrix games obtained combining two-player games associated to links that are not zero-sum, i.e., graphical polymatrix games whose zero-sum nature is a global rather than a local property.

In the paper [58], the authors investigate the problem of representing correlated equilibria in graphical games and they obtain fundamental representational and computational results. They show that it is possible to represent correlated equilibria of a graphical game (up to a notion of expected payoff equivalence) with the same succinctness of the game representation in terms of a suitable local Markov network, an undirected probabilistic graphical model defined over the same graph of the game. In doing this they draw a powerful connection between the two graphical models: on one side the graphical representation of a game encodes strategic dependencies by means of a graph and some local utility functions characterizing the interactions among neighboring players, on the other side the associated Markov network encodes on the same graph structure the probabilistic interactions among players in correlated equilibria via local potential functions, each associated to a neighborhood on the graph. More precisely, the considered local Markov networks are defined as probability distributions P over the game's configurations space $\mathcal{X}$ that can be factorized over the closed neighborhoods $\mathcal{N}_i^\bullet$ of the game's graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as

$$P(x) = \frac{1}{Z}\left(\Pi_{i\in\mathcal{V}}\varphi_i(x_{\mathcal{N}_i^\bullet})\right)$$

where the functions $\varphi_i : \mathcal{X}_{\mathcal{N}_i^\bullet} \to [0, \infty)$ are called local potentials and $Z = \sum_{x\in\mathcal{X}} \prod_{i\in\mathcal{V}} \varphi_i(x_{\mathcal{N}_i^\bullet})$ is a normalization factor. Then the aformationed result states that any correlated equilibrium P of a graphical game on a graph $\mathcal{G}$ can be represented as local Markov networks on $\mathcal{G}$ up to payoff equivalence, meaning that there exists a distribution Q, payoff equivalent to P, that is still a correleted equilibrium for the game and is a local Markov network on $\mathcal{G}$. This result is structural, in the sense that it is obtained as a consequence of the graphical structure of a game, and it is representational in the sense that it produces a representation of a game feature, namely correlated equilibria, which inherits the compactness and descriptive power of the graphical game formalism. Following the lines of this work [58], in Chapter 6 we will investigate the representation of correlated equilibria for separable games. A different result, but in the same spirit as the previous one, is obtained in [36] where the authors propose a mapping of any graphical game into a Markov random field, whose graph structure is not exactly the same but is closely related to that of the game, and show that the existence and structure of pure Nash equilibria of the game can be equivalently investigated by means of probabilistic queries on the corresponding Markov random field.

A further relationship between graphical games and probabilistic graphical models has been established in [14] for the class of graphical potential games. This work shows that potential functions of a graphical potential game correspond to positive Markov random fields with the same graph structure, i.e., to probability distributions P that can be factorized over the maximal cliques $\mathcal{C}\ell(\mathcal{G})$ of the game's graph $\mathcal{G}$ as

$$P(x) = \frac{1}{Z}\left(\prod_{C\in\mathcal{C}\ell}\varphi_C(x_C)\right)$$

where the positive functions $\varphi_C : \mathcal{X}_C \to (0,\infty)$ are called local potentials and $Z = \sum_{x\in\mathcal{X}}\prod_{i\in\mathcal{V}}\varphi_i(x_{\mathcal{N}_i})$ is a normalization factor. More precisely, to a potential function ϕ of a graphical game it corresponds a positive Markov random field defined by

$$P(x) = e^{\phi(x)} \tag{3}$$

(with the normalization assumption that $\sum_{x\in\mathcal{X}} e^{\phi(x)} = 1$). The authors actually show that the connection between potential functions of graphical games and Markov random fields is much stronger, as not only it is possible to map a potential function of the game into a positive Markov random field with the same graphicality property but also that each positive Markov random field over a graph correspond to a graphical potential game on that graph, resulting into an equivalence of graphical potential games and positive Markov random fields. This connection allows, in particular, to translate the factorization property of positive Markov random fields into a decomposition property for potential functions, deriving a structural result for graphical potential games: any potential function of a graphical potential game can be additively decomposed as

$$\phi(x) = \sum_{C\in\mathcal{C}\ell(\mathcal{G})}\phi_C(x_C)$$

into the sum of local potentials, each associated to a maximal clique of the game's graph and depending only on the actions of players belonging to that clique. A counterpart of this and further results are presented in [76] for the class of graphical transformed potential games defined therein. As discussed in Chapter 7, such results admit an extension and refinement in the setting of separable games.

In conclusion, the results collected in this brief summary demonstrate the power of the graphical games formalism in the analysis of structural properties of multi-player games. However, while there is already a large amount of literature focusing on specific graphical games (which we will present later on in this chapter), a general theory is still missing and in this sense the presented works represent remarkable exceptions. How does the graphicality of a game reflect on its properties is still largely unexplored.

1.3.2 *Algorithmic and computational results*

Computational game theory is the branch of game theory that is concerned with the development of procedures to perform game-theoretic computations. The inquiries that these procedures aim to support cover a wide range of questions that naturally arise from real systems that are modelled by games but the main focus is on identifying the many different notions of equilibria in games, which include pure and mixed Nash equilibria and correlated equilibria. The reason for the great interest in this quest is of twofold nature, both practical and conceptual. On the practical side, being able to obtain equilibria of a game is useful as a predictive and prescriptive tool, since many models assume that the systems under analysis have reached or will eventually reach a steady state, i.e., an equilibrium configuration. On the other hand, equilibria are standard solution concepts in game theory. If such solutions cannot be efficiently computed by a machine it would be unreasonable to expect a system of agents to realise them and the very validity of the notion would be lost.

A concise representation of the input data plays an essential role in the creation of efficient algorithmic procedures. Indeed, the size of the normal form games' representation is exponential in the number of players. This undermines the efficiency of algorithms operating on games, since even an algorithm with polynomial complexity results in an execution time which is exponential in the number of players. Network game representations provide the required compactness in many cases. Indeed, games modelling real world phenomena naturally possess some network structure reflecting the pattern of interactions among agents, which are usually local. In fact, the formalism of graphical games has made it possible to derive significant and provably efficient algorithms for the computation of both pure Nash [47, 36] and correlated equilibria [58, 80].

[47] addresses the problem of the existence and computation of pure Nash equilibria and strong Nash equilibria in games. The paper focuses on graphical and hypergraphical representations of games and shows that determining the existence of pure and strong Nash equilibria of a game is a hard problem even under the strong assumptions of bounded neighborhood or graph/hypergraph acyclicity on its graphical and hypergraphical representations respectively. However, the work reveals how network representations of games are the suitable language to identify and express reasonable restrictions that make the problem tractable, which are

the combination of two weaker assumptions, namely the small neighborhood of the graphical representation and bounded hypertree width of the hypergraphical representation. The same problem has been further investigated in [36] with a considerably different approach that leads to an improvement of the results of [47]. This work present a way of mapping any graphical game to a suitable Markov random field in such a way that the problem of assessing the existence of a pure Nash equilibrium of the game is reduced to the computation of a maximum a posteriori configuration of the Markov random field, a central task in the theory of graphical models for which efficient algorithms are known. The same reduction allows to obtain a compact description of all pure Nash equilibria of the game by extracting marginal probability distributions on the cliques of the corresponding graphical model, which can be achieved by means of the junction tree algorithm. This result produces a polynomial algorithm for computing pure Nash equilibria of graphical games without any acyclicity assumption, expanding the previously known class of tractable problems [47] to include graphical games with unbounded but slowly increasing tree-width and bounded neighborhood size.

In the context of mixed strategies, existence of mixed Nash equilibria is guaranteed [73] so that research focuses on the computational aspects of finding such equilibria. The problem of deciding whether a mixed Nash can be found in polynomial time has been studied in depth [68, 79], culminating in [38] where the problem was shown to be PPAD-complete both for normal form and for graphical games, so that with high confidence the answer is negative.

The above considerations concerning existence remain true for correlated equilibria but, in contrast, the question regarding the tractability of their computation admits a quite different answer. As an illustrative example we present some works that focus on this problem, which have the further merit of underlining the aforementioned connection between graphical games and probabilistic graphical models.

The computation of correlated equilibria for normal form games can be immediately reduced to the solution of a system of linear inequalities. Indeed, correlated equilibria P of a game u are characterized by non-negativity and normalization constraints, describing the fact that correlated equilibria are probability distributions

$$\forall x \in \mathcal{X}, P(x) \geq 0, \quad \sum_{x \in \mathcal{X}} P(x) = 1,$$

and by equilibrium constraints for each player $i \in \mathcal{V}$

$$\forall x_i, y_i \in \mathcal{A}_i, \quad \mathbb{E}_{x \sim P|x_i} u_i(x_i, x_{-i}) \geq \mathbb{E}_{x \sim P|x_i} u_i(y_i, x_{-i})$$

stating that no player i has any incentive (measured with expected payoff gain) in unilaterally deviating from the action prescribed by the correlated equilibrium. These are in fact linear constraints, if restated in the following equivalent form:

$$\forall x_i, y_i \in \mathcal{A}_i, \qquad \sum_{x_{-i} \in \mathcal{X}_{-i}} u_i(x_i, x_{-i}) P(x_i, x_{-i}) \geq \sum_{x_{-i} \in \mathcal{X}_{-i}} u_i(y_i, x_{-i}) P(x_i, x_{-i}). \tag{4}$$

This fact allows to define a linear programming problem whose feasible set coincides with correlated equilibria of the game and that is able to select specific correlated equilibria depending on the chosen objective function. The resulting optimization problem has the probability values $P(x)$ for all configurations $x \in \mathcal{X}$ as optimization variables, so that the number of variables is $|\mathcal{X}|$. The number of constraints is also polynomial in $|\mathcal{X}|$, due to the non-negativity constraints. This, together with the fact that linear programs belong to the polynomial complexity class, implies that correlated equilibria can be computed in time which is polynomial in $|\mathcal{X}|$, and consequently polynomial in the normal form representation of the game (whose size is $n|\mathcal{X}|$). This result is only apparently satisfactory, as the complexity of the problem is polynomial only when measured with respect to the verbose normal form representation of the input game, which is in turn exponential in the graphical representation of the same game whenever, for example, the graph structure of the game has bounded degree. This reasoning shows that a more significant way of assessing the quality of a computational procedure for graphical games is measuring its complexity with respect to the network representation of the input. This general consideration poses a more difficult but more interesting challenge in the context of correlated equilibria computation, which was first tackled in [58]. The authors employ the representational results described in the previous section to restate the above linear programming in local form. This approach involves local probability optimization variables $P_i(x_{\mathcal{N}_i^\bullet})$ and exploits the locality of utility functions to express the equilibrium conditions (4) in local form as:

$$\forall x_i, y_i \in \mathcal{A}_i, \qquad \sum_{x_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i}} u_i(x_i, x_{\mathcal{N}_i}) P_i(x_i, x_{\mathcal{N}_i}) \geq \sum_{x_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i}} u_i(y_i, x_{\mathcal{N}_i}) P_i(x_i, x_{\mathcal{N}_i}).$$

Local probability variables are again subject to non-negativity and normalization constraints,

$$\forall x_{\mathcal{N}_i^\bullet} \in \mathcal{X}_{\mathcal{N}_i^\bullet}, P_i(x_{\mathcal{N}_i^\bullet}) \geq 0, \qquad \sum_{x_{\mathcal{N}_i^\bullet} \in \mathcal{X}} P_i(x_{\mathcal{N}_i^\bullet}) = 1,$$

and to "Intersection Consistency Constraints", which couple local probabilities to account for the possible overlap of different neighborhoods in the underlying graph. More precisely, for each i and j in $\mathcal{V}$ and for each $y \in \mathcal{X}_{\mathcal{N}_i^\bullet \cap \mathcal{N}_j^\bullet}$

$$\sum_{x_{\mathcal{N}_i^\bullet} \in \mathcal{X}_{\mathcal{N}_i^\bullet} : x_{\mathcal{N}_i^\bullet \cap \mathcal{N}_j^\bullet} = y} P_i(x_{\mathcal{N}_i^\bullet}) = \sum_{x_{\mathcal{N}_j^\bullet} \in \mathcal{X}_{\mathcal{N}_j^\bullet} : \mathcal{N}_i^\bullet \cap \mathcal{N}_j^\bullet = y} P_j(x_{\mathcal{N}_j^\bullet}).$$

This system of linear inequalities can be solved in linear time in the graphical representation of the game. Taking advantage of the equivalence of correlated equilibria of graphical games to local Markov networks with the same graph structure, the authors show that each equilibrium of the game is represented via its local marginals in the solution set. Viceversa, each solution of the above system can be mapped to a correlated equilibrium of the game provided that the graph of the game is a tree. This result has the merit of deriving a polynomial time algorithm for computing correlated equilibria of graphical games relying explicitly on the graphical structure but its applicability is limited to tree graph structures. Such limitation is overcome in [80], where the previous result is extended to graphical games of bounded tree-width. More in general, [80] proposes polynomial-time algorithms for finding correlated equilibria in multiplayer games that admit a suitable compact representation, formally defined therein as a "succinct representation". This very general notion includes as special cases polymatrix games, graphical games and hypergraphical games, as well as several other game models. Due to the wide scope of this result, which is not limited to network representations of games, its derivation does not exploit the network structure of games directly. Instead, it exploits the succinctness property of the game's representation at its best, resulting in a very strong result with broad applicability.

1.3.3 *Specific network game models*

As previously mentioned, one of the most fundamental issues related to the normal form representation of games is that it ignores the underlying structure that the game may possess. Indeed, a lot of problems that are modelled using game theory naturally possess some network structure, of which the following are just a few examples. Organizational structures are usually described by tree-like networks, where agents located at a certain level are engaged in games with their supervisors, located at higher levels. In computer networks, computer machines are located at nodes and negotiate with neighbouring machines to balance the work load. Perhaps the most famous example of naturally emerging network structure are social structures. They are described by very complex networks and it is well known that the structure of such networks is responsible for the emergence of collective behaviours. For example, the evolution of conventions has been shown to result from the repeated resolution of local conflicts. Finally, there are physical structures, such as the ones describing the distribution of particles in space. Many models in statistical mechanics, such as the well-known Ising model, assume that particles are spatially distributed at the nodes of a lattice and they are able to describe the emergence of macroscopic behaviours as the result of microscopic interactions between particles, due to some short range physical law. All these network structures, and many others, are lost if we use a tabular representation for games.

For this reason network representations of games have recently emerged as a unified framework for modelling interactions in many social and economic settings [54, 42, 52, 27]. They allow one to analyse the emergence of phenomena such as peer effects, technology adoption, spread of ideas and innovation [87, 70], propagation of shocks or perturbations [4], consensus formation [71, 25], diffusion of crime or education [16], and commitment to public good [26, 5].

There exists a very rich literature that investigates specific game models, of which the cited works are just a small portion. From the point of view of our discussion, we are most interested in highlighting the common traits to the different approaches. When focusing on a specific game model, the aim is usually to derive a characterization of interesting game properties in terms of particular features of the underlying network. To describe this process we present some significant examples.

Network coordination games are popular game models [52] with a variety of applications in economics, social sciences, and biology. In their simplest version, they are mathematically described as strategic games with binary action set, where players are interconnected through a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Specifically, when the binary actions are $\mathcal{A} = \{0, 1\}$, the utility of a player in a network coordination game is an affine increasing function of the number of their neighbors in the network playing the same action. A special instance of such games is obtained letting

$$u_i(x_i, x_{-i}) = \sum_{j \in \mathcal{V}} W_{ij} \big((1-r) x_i x_j + r(1-x_i)(1-x_j) \big) , \tag{5}$$

where r in $(0, 1)$ is a parameter. This utility function describes the fact that each agent i receives a reward from the interaction with each of their neighbors. More precisely, they receive a different reward for each match on action 1 or action 0 with a neighbor j, which depends on the weight W_{ij} of the link connecting them and on the parameter r. Notice that the game can be represented as graphical polymatrix game on $\mathcal{G}$, as the utilities of players are clearly pairwise separable. The common value r is referred to as the threshold value of the game because the above payoff structure induces a threhold behavior for the best response corerspondence, which is the function that gives, for each player, their best actions to play given the configuration of all the other players. In particular, the best response for a player in the coordination game is 1 if the total weight of links to their neighbors playing 1 is above a fraction r of their degree in $\mathcal{G}$. How does the network structure determine the properties of such games has been extensively investigated in the literature from many different perspectives.

A notable example is given by [71], which considers an infinite population of locally interacting agents, and [54], which adapts the results to a finite population setting. It is immediate to see that, irrespectively of the graph structure, the network coordination game admits two pure Nash equilibria, corresponding to the two configurations where all players agree on one of the two available actions, also referred to as consensus configurations. The two papers then tackle some more elaborate questions concerning, for example, the existence of co-existent equilibria, where both actions are concurrently represented. Interestingly, this question proves to admit an answer in terms of the network structure, and, specifically, on the graph topological notion of cohesiveness. Precisely, a set $\mathcal{S} \subset \mathcal{V}$ of nodes of a graph $\mathcal{G}$ is said to be r-cohesive if each node in $\mathcal{S}$ has at least a fraction r of its degree deriving from links pointing to $\mathcal{S}$. Cohesiveness measures how tight a sub-community is in a network and allows different groups of players in the network to sustain different behaviors. Indeed, a coexistent configuration x where players in a set $\mathcal{S} \subset \mathcal{V}$ play action 1 and the remaining players $\mathcal{V} \setminus \mathcal{S}$ play action 0 is a Nash equilibrium for the network coordination game if and only if $\mathcal{S}$ form an r-cohesive set in $\mathcal{G}$ while the set $\mathcal{V} \setminus \mathcal{S}$ is $(1-r)$-cohesive. The usefulness of the cohesiveness notion is not limited to the mentioned application. On the contrary, it is both the basis for further investigations on the structural properties of network coordination games, such as the identification of atomic behavioral communities in networks composed of agents that always behave the same way in all equilibria of coordination games [53], and it provides an inspiration to extend similar ideas beyond network coordination games [10, 11].

[81] proposes synchronisation games on hypergraphs, that generalize graphical polymatrix coordination and anti-coordination games to the hypergraphical setting. Synchronization games are games on a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ where only hyperlinks where a consensus on some action is realized contribute to the utility of players. More precisely, each hyperlink $\mathcal{J} \in \mathcal{L}$ is associated with a possibly different weight function $w_{\mathcal{J}}$ defined on the common action set of players $\mathcal{A}$ and the utility of players is defined as

$$u_i(x) = \sum_{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}} \bar{w}_{\mathcal{J}}(x), \quad \forall x \in \mathcal{X}$$

where

$$\bar{w}_{\mathcal{J}}(x) = \begin{cases} 0 & \text{if } \exists i, j \in \mathcal{J} : x_i \neq x_j \\ w_{\mathcal{J}}(a) & \text{if } \forall i \in \mathcal{J}, x_i = a \in \mathcal{A}. \end{cases}$$

In words, in a configuration x of the game, each player receives no utility from any hyperlink $\mathcal{J} \in \mathcal{L}$ where the restricted configuration is not uniform, i.e., such that $\exists i, j \in \mathcal{J}$ with $x_i \neq x_j$. Instead, the player receives a reward from each hyperlink $\mathcal{J} \in \mathcal{L}$ where a convention is established, i.e., such that $x_{\mathcal{J}} = a\mathbb{1}_{\mathcal{J}}$ for some $a \in \mathcal{A}$. When all weight functions are positive, the game is referred to as coordination games on hypergraphs and it captures a weak form of high-order coordination, where only full consensus in communities represented by hyperlinks is rewarded. When, instead, all weights are negative the game models a weak form of high-order mis-coordination, where players have an incentive in deviating only if all other members of the hyperlink play a common action. The authors derive sufficient conditions for the existence and the efficient computability of strong equilibria for the game based on some acyclicity properties of the underlying hypergraph.

A similar class of hypergraphical games that model a wider range of high-order mis-coordination behaviors are hypergraphical clustering games of mis-coordination [82]. In these games players may exhibit different levels of homogeneity-aversion. Players with weak homogeneity-aversion receive a reward from every hyperlink they belong to in which there is at least one player with a different action. Players with a strong homogeneity-aversion, instead, behave as in the previous game model [81] and are only rewarded for mis-coordinating with all members of an hyperlink. The authors study the price of anarchy of such games, which measures the inefficiency of pure Nash equilibria with respect to a system optimum configuration, deriving bounds that depend on the size of the hypergraph.

1.4 ORGANIZATION OF THE MANUSCRIPT

The thesis is organized as follows.

- In Chapter 2, we introduce the main mathematical tools and notation that will turn useful throughout the thesis. In Section 2.2 we present some central graph-theoretic notion, starting from graphs up to undirected and forward directed hypergraphs, and we describe the relation among these concepts. Section 2.3 focuses on decomposable hypergraphs and on the special properties of probabilistic models endowed with a decomposable hypergraph structure that are at the base of the results of Chapter 6. Then, in Section

2.4 we present the basics of game theory, with a particular focus on graphical games, and we derive a relevant decomposition of the space of games into potential, harmonic and normalized games. Finally, in Section 2.5 we introduce Markov Random Fields, a special family of probabilistic graphical models with strong connections to the theory of potential games.

- In Chapter 3, we introduce separable games and show how interactions in a game can be effectively described by means of forward directed hypergraphs.
- In Chapter 4, we tackle a fundamental question about the separable representation of games, that is the existence of a minimal separable representation for games, showing that each game possesses a minimal forward directed hypergraph describing its separability. We also discuss in details the relation between separability and the notion of strategic equivalence of games, expanding on its implications on the computational complexity of the separable representation.
- In Chapter 5, we change our point of view: instead of focusing on investigating the separability of a given game, we describe how to obtain the most accurate representation of it with a prescribed separability property by means of projections. This framework allows to obtain compact approximate representation of games that can be exploited to ease the analysis of games properties. We give an example of this approach in Section 5.2.

We then turn to describe the way the underlying hypergraphical structure of separable games reflects on their properties. In particular, we focus on the structural properties of correlated equilibria of separable games and of the potential function of separable potential games by deriving extensions to the separable setting of relevant corresponding results for graphical games [58, 14]. In both cases we show that the entities of interest can be decomposed according to the hypergraphical structure of the separable game.

- In Chapter 6 we show that correlated equilibria of separable games can be factorized over hyperlinks of the underlying hypergraph, up to a notion of equivalence that again is based on the hypergraph structure.
- Similarly, in Chapter 7 we show how the potential function of separable potential games can be additively decomposed over hyperlinks of the underlying game hypergraph. We comment on the connections of this result with

the theory of Markov Random Fields and we describe some of its several implications, both on the analysis of better response paths of separable potential games and on the interplay of separability with the potential-harmonic decomposition of games.

- Finally, in Chapter 8 we give a geometric characterization of separability deriving checkable conditions to identify the minimal hypergraph of a game. We then describe an algorithm to find such minimal hypergraph and we propose an implementation of it. We extend this analysis to the case where the game is not fully known but only partial data about the game are available. For this setting we propose a procedure to construct hypergraphs that are compatible with a partial game specification.

- We conclude the thesis with Chapter 9 where we summarize the main contributions of this work and present open problems that represent possible directions for future research.

To promote the consistency of the presentation we choose not to include in the present manuscript all the products of the PhD research. More precisely, the content of this thesis includes (but is not limited to) the two works [8] and [9]. Instead, we do not include the following works:

- [84] studies Nash equilibria for games where a mixture of coordinating and anti-coordinating agents, with possibly heterogeneous thresholds, coexist and interact through an all-to-all network. Whilst games with only coordinating or only anti-coordinating agents are potential, also in the presence of heterogeneities, this does not hold when both type of agents are simultaneously present. This makes their analysis more difficult and existence of Nash equilibria not guaranteed. Our main result is a checkable condition on the threshold distributions that characterizes the existence of Nash equilibria in such mixed games. When this condition is satisfied an explicit algorithm allows to determine the complete set of such equilibria. Moreover, for the special case when only one type of agents is present (either coordinating or anti-coordinating), our results allow an explicit computation of the cardinality of Nash equilibria.

- In [10] we consider network games with coexisting coordinating and anti-coordinating players. We first provide graph-theoretic conditions for the existence of pure-strategy Nash equilibria in mixed network coordination/anti-coordination games of arbitrary size. For the case where such conditions are met, we then study the asymptotic behavior of best-response dynamics and provide sufficient conditions for finite-time convergence to the set of Nash equilibria. Our results build on an extension and refinement of the notion of network cohesiveness and on the formulation of the new concept of network indecomposibility.

- In [11] we study the robustness of binary-action heterogeneous network coordination games equipped with an external field modeling the different players' biases towards one action with respect to the other one. We prove necessary and sufficient conditions for global stability of consensus equilibria under best response type dynamics, robustly with respect to constant or time-varying values of the external field. We then apply these results to the analysis of mixed network coordination and anti-coordination games and find sufficient conditions for existence and global stability of pure strategy Nash equilibria. Our results apply to general weighted directed interaction networks and build on supermodularity properties of coordination games in order to characterize conditions for the existence of a novel notion of robust improvement and best response paths.

2

MATHEMATICAL CONCEPTS AND TOOLS

In this chapter we are going to introduce the notation, theoretical concepts and tools that will be used throughout this thesis. More in detail, we present notions from graph and hypergraph theory, from game theory and from the theory of graphical models that will be used to build and analyse the separable game model in the following chapters.

2.1 BASIC NOTATION

We start by presenting the basic notation to be used throughout this work. Vectors are denoted with lower case, matrices and random variables with upper case, and sets (and set-valued functions) with calligraphic letters. A subscript associated to vectors, for instance $v_{\mathcal{A}}$, represents the sub-vector that is the restriction of a vector v in $\mathbb{R}^n$ on the set of indices $\mathcal{A} \subseteq \{1, 2, \ldots, n\}$. We indicate with $\mathbb{1}$ the all-1 vector, regardless of its dimension, and with I the identity operator.

We denote by $\Delta(\mathcal{X})$ the space of probability distributions $P : \mathcal{X} \to [0, 1]$ over a given set $\mathcal{X}$. For a probability distribution $P : \mathcal{X} \to [0, 1]$ we denote by $P(x_{\mathcal{J}})$ the marginal probability associated to the set $\mathcal{J} \subset \mathcal{V}$:

$$P(x_{\mathcal{J}}) = \sum_{\substack{y \in \mathcal{X} \\ y_{\mathcal{J}} = x_{\mathcal{J}}}} P(y) \tag{6}$$

The expected value of a random variable $F : \mathcal{X} \to \mathbb{R}$ will be denoted by

$$\mathbb{E}_{x \sim P} F(x) = \sum_{x \in \mathcal{X}} F(x) P(x).$$

2.2 GRAPH-THEORETIC PRELIMINARIES

In this section we introduce some basic graph-theoretic definitions and notation. We shall begin with directed graphs, then consider undirected and forward directed hypergraphs, and finally show how these notions are related.

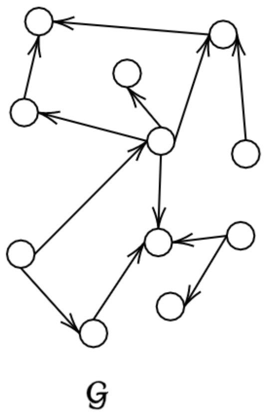


Figure 1: A directed graph $\mathcal{G}$.

2.2.1 *Directed graphs*

Definition 2.2.1. *A* directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is the pair of a finite node set* $\mathcal{V}$ *and of a link set* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$*, where a link* (i, j) *in* $\mathcal{E}$ *is meant as directed from its tail node* i *to its head node* j*.*

Throughout this work, we shall consider directed graphs containing no self-loops, i.e., such that (i, i) does not belong to the link set $\mathcal{E}$ for any node i in $\mathcal{V}$, and refer to them simply as graphs. An example of directed graph is shown in Figure 1.

We shall denote by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ and $\mathcal{N}_i^\bullet = \mathcal{N}_i \cup \{i\}$ the open and, respectively, closed out-neighborhoods of a node i in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The set of parents of a node i, denoted with $\mathrm{pa}(i)$, is defined as the set of nodes

$$\mathrm{pa}(i) = \{j \in \mathcal{V}, (j, i) \in \mathcal{E}\}, \tag{7}$$

while the open out neighborhood of node i is also referred to as the set of i's children $\mathrm{ch}(i) \equiv \mathcal{N}_i$. The intersection of two graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ with the same node set $\mathcal{V}$ is the graph $\mathcal{G}_1 \sqcap \mathcal{G}_2 = (\mathcal{V}, \mathcal{E})$ with link set $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$. We shall say that $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ is a subgraph of $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ and write $\mathcal{G}_1 \preceq \mathcal{G}_2$, if $\mathcal{E}_1 \subseteq \mathcal{E}_2$, equivalently, if $\mathcal{G}_1 \sqcap \mathcal{G}_2 = \mathcal{G}_1$. The product of two graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ is the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}) = \mathcal{G}_1 \otimes \mathcal{G}_2$ such that $\mathcal{V}$ equals the cartesian product $\mathcal{V}_1 \times \mathcal{V}_2$ while $\mathcal{E} = \{((i_1, i_2), (j_1, i_2)) : (i_1, j_1) \in \mathcal{E}_1\} \cup \{((i_1, i_2), (i_1, j_2)) : (i_2, j_2) \in \mathcal{E}_2\}$.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a set $\mathcal{A} \subset \mathcal{V}$, the induced graph $\mathcal{G}_\mathcal{A}$ is defined as $(\mathcal{A}, \mathcal{E}_\mathcal{A})$ where $\mathcal{E}_\mathcal{A} = \{(i, j) \in \mathcal{E} : i, j \in \mathcal{A}\}$.

We shall consider undirected graphs as a special case of graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that there exists a directed link (i, j) in $\mathcal{E}$ if and only if the reversed directed link (j, i) in $\mathcal{E}$ exists as well.

A path of length ℓ on a graph $\mathcal{G}$ is a sequence of nodes $(i_k)_{k=0}^{\ell}$ such that each couple of consecutive nodes is connected by a link following the direction of increasing indices, i.e., for each $k \in \{0, \dots, \ell - 1\}$, $(i_k, i_{k+1}) \in \mathcal{E}$. A path is closed if its starting and ending nodes coincide, i.e., if $i_0 = i_\ell$. A closed path is called a cycle. A related notion is that of chain on a graph. A chain of length ℓ on a graph $\mathcal{G}$ is a sequence of distinct nodes $(i_k)_{k=0}^{\ell}$ such that each couple of consecutive nodes is connected by a single link whose direction may match or be opposite to that of increasing indices, i.e., for each $k \in \{0, \dots, \ell - 1\}$, either $(i_k, i_{k+1}) \in \mathcal{E}$ or $(i_{k+1}, i_k) \in \mathcal{E}$.

For two nodes $i, j \in \mathcal{V}$ of an undirected graph we denote by $\Delta(i, j)$ the distance between the two nodes in the graph $\mathcal{G}$, defined as the length of any shortest path between i and j. We will make use of the following property of undirected graphs, which can be expressed in terms of paths.

Definition 2.2.2 (Triangulated graph)**.** *A triangulated graph, also called a decomposable or chordal graph (see [64]), is an undirected graph with the property that every cycle of length $n \geq 4$ possesses a chord, i.e., two non consecutive vertices that are neighbours.*

Certain supergraphs of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ will play a relevant role in our analysis: they are all undirected and obtained by keeping the same node set $\mathcal{V}$ and augmenting the link set $\mathcal{E}$ as follows. First, let $\mathcal{G}^{\leftrightarrow} = (\mathcal{V}, \mathcal{E}^{\leftrightarrow})$ be the minimal undirected supergraph of $\mathcal{G}$. Clearly, $\mathcal{G}^{\leftrightarrow}$ is obtained from $\mathcal{G}$ by making all its links undirected, i.e., it has link set $\mathcal{E}^{\leftrightarrow} = \mathcal{E} \cup \{(j, i) : (i, j) \in \mathcal{E}\}$. Moreover, let $\mathcal{G}^{\triangle} = (\mathcal{V}, \mathcal{E}^{\triangle})$ be the graph obtained from $\mathcal{G}^{\leftrightarrow}$ by adding links between out-neighbors $\mathcal{N}_i$ of every node in $\mathcal{G}$, i.e., $\mathcal{G}^{\triangle}$ has link set

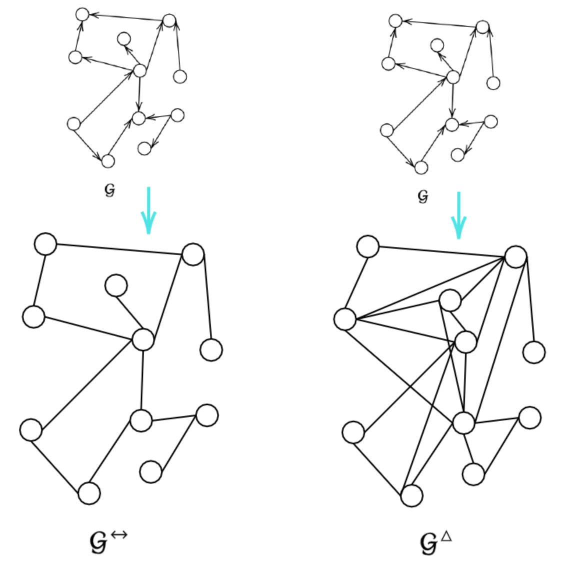
$$\mathcal{E}^{\triangle} = \mathcal{E}^{\leftrightarrow} \cup \bigcup_{i \in \mathcal{V}} \{(j, l) : j, l \in \mathcal{N}_i, \, j \neq l\} \, .$$

Equivalently, $\mathcal{G}^{\triangle}$ can be characterized as the graph having neighborhoods $\mathcal{N}_i^{\triangle}$ for each player $i \in \mathcal{V}$ given by

$$\mathcal{N}_i^{\triangle} = \bigcup \{\mathcal{N}_h^{\bullet} \setminus \{i\} : h \in \mathcal{V}, i \in \mathcal{N}_h^{\bullet}\} \, . \tag{8}$$

The graphs $\mathcal{G}^{\leftrightarrow}$ and $\mathcal{G}^{\triangle}$ for $\mathcal{G}$ as in Figure 1 are shown in Figure 2.

A clique is a completely connected subset of nodes $\mathcal{C} \subset \mathcal{V}$, i.e., a set of nodes such that for every couple of nodes $i, j \in \mathcal{C}$, $(i, j) \in \mathcal{E}$. The size $h = |\mathcal{C}|$ of a clique $\mathcal{C}$ is the number of nodes it is composed of; a clique of size h is often denoted



(a) Construction of $\mathcal{G}^{\leftrightarrow}$.

(b) Construction of $\mathcal{G}^{\triangle}$.

Figure 2: Supergraphs of the graph $\mathcal{G}$ represented in Figure 1.

as a h-clique. A clique $\mathcal{C}$ is said to be maximal if it is not contained in any larger size clique, i.e., if there is no clique $\tilde{\mathcal{C}}$ of $\mathcal{G}$ such that $\mathcal{C} \subsetneq \tilde{\mathcal{C}}$. Given a graph $\mathcal{G}$, we will denote by $\mathcal{C}\ell(\mathcal{G})$ the family of its maximal cliques. Figure 3 shows an example of a graph and highlights its cliques. The light blue 3-cliques and the dark blue 4-cliques form maximal cliques.

2.2.2 *Hypergraphs*

Definition 2.2.3. *A* hypergraph *(shortly, a* H-graph*) is the pair $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ of a finite node set $\mathcal{V}$ and of a set $\mathcal{L}$ of undirected hyperlinks, each of which is a nonempty subset of nodes [28].*

A simple way to visualize hypergraphs is by identifying each hyperlink with a colour or pattern and by grouping/filling nodes with the colours and pattern corresponding to hyperlinks they belong to. An example of this visualization is in Figure 4 where nodes coincide with nodes $\mathcal{V}$ of the graph $\mathcal{G}$ from Figure 1 while hyperlinks coincide with closed neighbourhoods of the same graph.

A H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is referred to as *simple* if no undirected hyperlink $\mathcal{J}$ in $\mathcal{L}$ is strictly contained in another undirected hyperlink $\mathcal{K}$ in $\mathcal{L}$. The simple H-

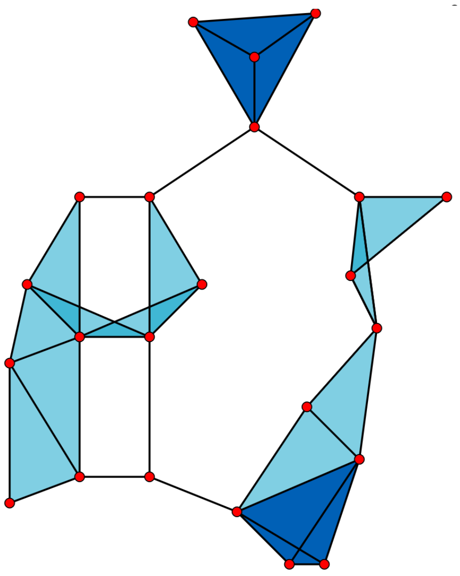


Figure 3: A graph with its cliques (from https://en.wikipedia.org/wiki/Clique_(graph_theory)). Links of the graph are 2-cliques, 3-cliques are represented as light and dark blue triangles, while 4-cliques correspond to dark blue areas.

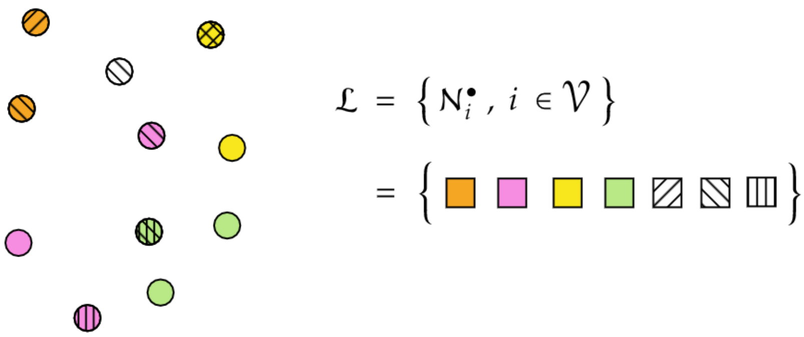


Figure 4: Representation of an H-graph. Nodes $\mathcal{V}$ and hyperlinks $\mathcal{L}$ coincide with nodes and closed neighbourhoods of the graph from Figure 1.

graph associated to a H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is the H-graph $\overline{\mathcal{H}} = (\mathcal{V}, \overline{\mathcal{L}})$ with set of undirected hyperlinks

$$\overline{\mathcal{L}} = \{\mathcal{J} \in \mathcal{L} : \nexists \mathcal{K} \in \mathcal{L} \text{ s.t. } \mathcal{K} \supsetneq \mathcal{J}\}.$$

We shall say that a H-graph $\mathcal{H}_1 = (\mathcal{V}, \mathcal{L}_1)$ is a sub-H-graph of another H-graph $\mathcal{H}_2 = (\mathcal{V}, \mathcal{L}_2)$ and write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if, for every undirected hyperlink $\mathcal{J}_1$ in $\mathcal{L}_1$, there exists some undirected hyperlink $\mathcal{J}_2$ in $\mathcal{L}_2$ such that $\mathcal{J}_1 \subseteq \mathcal{J}_2$. Notice that both $\mathcal{H} \preceq \overline{\mathcal{H}}$ and $\overline{\mathcal{H}} \preceq \mathcal{H}$ and that in fact $\overline{\mathcal{H}}$ is the only simple H-graph with this property. Also, observe that $\mathcal{H}_1 \preceq \mathcal{H}_2$ if and only if $\overline{\mathcal{H}_1} \preceq \overline{\mathcal{H}_2}$.

Given two H-graphs $\mathcal{H}_1 = (\mathcal{V}, \mathcal{L}_1)$ and $\mathcal{H}_2 = (\mathcal{V}, \mathcal{L}_2)$ with the same node set $\mathcal{V}$, the intersection of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the H-graph $\mathcal{H}_1 \sqcap \mathcal{H}_2 = (\mathcal{V}, \mathcal{L})$ with set of undirected hyperlinks

$$\mathcal{L} = \{\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2 \,:\, \mathcal{J}_1 \in \mathcal{L}_1, \, \mathcal{J}_2 \in \mathcal{L}_2\}.$$

The union of $\mathcal{H}_1$ and $\mathcal{H}_2$ is instead defined as the H-graph $\mathcal{H}_1 \sqcup \mathcal{H}_2 = (\mathcal{V}, \mathcal{L}_1 \cup \mathcal{L}_2)$.

Given a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ and a set $\mathcal{A} \subset \mathcal{V}$, the induced hypergraph $\mathcal{H}_\mathcal{A}$ is defined as $(\mathcal{A}, \mathcal{L}_\mathcal{A})$ where $\mathcal{L}_\mathcal{A} = \{\mathcal{J} \cap \mathcal{A} : \mathcal{J} \in \mathcal{L}\}$.

A H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is said to be *completely connected* if $\mathcal{V} \in \mathcal{L}$.

2.2.3 *Forward directed hypergraphs*

The following is a generalization of both notions of graphs and H-graphs given above.

Definition 2.2.4. *A* forward directed hypergraph *(*FDH-graph*) [45] is the pair $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ of a finite node set $\mathcal{V}$ and of a finite hyperlink set $\mathcal{D}$, where each hyperlink $d = (i, \mathcal{J})$ in $\mathcal{D}$ is the ordered pair of a node i in $\mathcal{V}$ (to be referred to as its tail node) and a nonempty subset of head nodes $\mathcal{J} \subseteq \mathcal{V} \setminus \{i\}$ (to be referred to as the hyperlink's head set).*

A simple way to visualize FDH-graphs is to identify each directed hyperlink with a colour or pattern. Then, each tail node can be connected to its hyperlink's head sets with some arrows. An example of this visualization is in Figure 5 where nodes coincide with nodes $\mathcal{V}$ of $\mathcal{G}$ from Figure 1. Each hyperlink connects a node to its open neighbourhood in the same graph $\mathcal{G}$.

A FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ is simple if for every hyperlink $(i, \mathcal{J})$ in $\mathcal{D}$, there exists no other hyperlink $(i, \mathcal{K})$ in $\mathcal{D}$ such that $\mathcal{J} \subsetneq \mathcal{K}$. The simple FDH-graph associated to a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ is the FDH-graph $\overline{\mathcal{F}} = (\mathcal{V}, \overline{\mathcal{D}})$ with hyperlink set

$$\overline{\mathcal{D}} = \{(i, \mathcal{J}) \in \mathcal{D} : \nexists (i, \mathcal{K}) \in \mathcal{D} \text{ s.t. } \mathcal{K} \supsetneq \mathcal{J}\} \subseteq \mathcal{D}$$

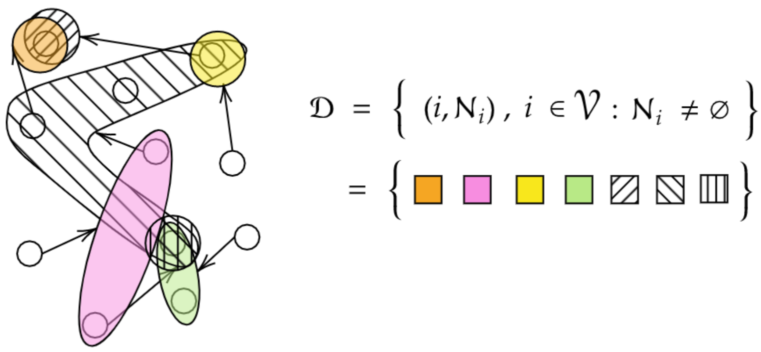


Figure 5: Visualization of an FDH-graph. Nodes correspond to nodes of the graph $\mathcal{G}$ of Figure 1; each node is the tail node of an hyperlink whose head set is the node's open neighborhood $\mathcal{N}_i$ in $\mathcal{G}$.

containing only hyperlinks in $\mathcal{D}$ with maximal head node set.

We shall say that a FDH-graph $\mathcal{F}_1 = (\mathcal{V}, \mathcal{D}_1)$ is a sub-FDH-graph of another FDH-graph $\mathcal{F}_2 = (\mathcal{V}, \mathcal{D}_2)$ and write $\mathcal{F}_1 \preceq \mathcal{F}_2$ if for every hyperlink $(i, \mathcal{J}_1)$ in $\mathcal{D}_1$ there exists some hyperlink $(i, \mathcal{J}_2)$ in $\mathcal{D}_2$ such that $\mathcal{J}_1 \subseteq \mathcal{J}_2$. Notice that both $\mathcal{F} \preceq \overline{\mathcal{F}}$ and $\overline{\mathcal{F}} \preceq \mathcal{F}$ and that in fact $\overline{\mathcal{F}}$ is the only FDH-graph with this property. Also, observe that $\mathcal{F}_1 \preceq \mathcal{F}_2$ if and only if $\overline{\mathcal{F}_1} \preceq \overline{\mathcal{F}_2}$.

The intersection of two FDH-graphs $\mathcal{F}_1 = (\mathcal{V}, \mathcal{D}_1)$ and $\mathcal{F}_2 = (\mathcal{V}, \mathcal{D}_2)$ with the same node set $\mathcal{V}$ is the FDH-graph $\mathcal{F}_1 \sqcap \mathcal{F}_2 = (\mathcal{V}, \mathcal{D})$ with set of directed hyperlinks

$$\mathcal{D} = \{(i, \mathcal{J}) : \exists (i, \mathcal{J}_1) \in \mathcal{D}_1, (i, \mathcal{J}_2) \in \mathcal{D}_2 \text{ s.t. } \mathcal{J}_1 \cap \mathcal{J}_2 = \mathcal{J}\}\,. \tag{9}$$

2.2.4 *How the various concepts are related*

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can naturally be identified with a FDH-graph on $\mathcal{V}$ whose hyperlinks are the original links in the graph (i, j) interpreted as $(i, \{j\})$; with slight abuse of notation in the following we may identify such FDH-graph with the original graph $\mathcal{G}$.

There is another natural way of relating directed graphs and FDH-graphs, that will play a key role in the following. On the one hand, to a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ we can associate the FDH-graph

$$\mathcal{F}^{\mathcal{G}} = (\mathcal{V}, \mathcal{D}^{\mathcal{G}}), \qquad \mathcal{D}^{\mathcal{G}} = \{(i, \mathcal{N}_i) \,|\, i \in \mathcal{V}\}\,. \tag{10}$$

As an example, Figure 5 shows the FDH-graph $\mathcal{F}^{\mathcal{G}}$ for the graph $\mathcal{G}$ of Figure 1. On the other hand, for a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ we can consider the directed graph

$$\mathcal{G}^{\mathcal{F}} = (\mathcal{V}, \mathcal{E}^{\mathcal{F}}), \qquad \mathcal{E}^{\mathcal{F}} = \{(i, j) \,|\, \exists (i, \mathcal{J}) \in \mathcal{D},\ j \in \mathcal{J}\}\,. \tag{11}$$

Observe that the following relations hold true:

$$\mathcal{G} = \mathcal{G}^{(\mathcal{F}^{\mathcal{G}})}, \quad \mathcal{F} \preceq \mathcal{F}^{(\mathcal{G}^{\mathcal{F}})}. \tag{12}$$

Analogously, on the one hand to every H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ we can associate the FDH-graph

$$\mathcal{F}^{\mathcal{H}} = (\mathcal{V}, \mathcal{D}^{\mathcal{H}}), \qquad \mathcal{D}^{\mathcal{H}} = \{(i, \mathcal{J}) \,|\, i \in \mathcal{V} \setminus \mathcal{J},\ \{i\} \cup \mathcal{J} \in \mathcal{L}\}, \tag{13}$$

while on the other hand to every FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ we can associate the H-graph

$$\mathcal{H}^{\mathcal{F}} = (\mathcal{V}, \mathcal{L}^{\mathcal{F}}), \quad \mathcal{L}^{\mathcal{F}} = \{\{i\} \cup \mathcal{J} :\ (i, \mathcal{J}) \in \mathcal{D}\}, \tag{14}$$

so that the following relations hold true:

$$\mathcal{H} = \mathcal{H}^{(\mathcal{F}^{\mathcal{H}})}, \quad \mathcal{F} \preceq \mathcal{F}^{(\mathcal{H}^{\mathcal{F}})}. \tag{15}$$

FDH-graphs $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ that are derived from H-graphs in the sense that $\mathcal{F} = \mathcal{F}^{\mathcal{H}}$ for some H-graph $\mathcal{H}$ are referred to as undirected as they are characterized by the property that

$$(i, \mathcal{J}) \in \mathcal{D} \ \Rightarrow\ (j, \{i\} \cup \mathcal{J} \setminus \{j\}) \in \mathcal{D}, \quad \forall j \in \mathcal{J}.$$

The FDH-graph $\mathcal{F}^{(\mathcal{H}^{\mathcal{F}})}$ is called the undirected FDH graph associated with $\mathcal{F}$ and, for notational simplicity, it is denoted by $\mathcal{F}^{\leftrightarrow} = (\mathcal{V}, \mathcal{D}^{\leftrightarrow})$. The set of its directed hyperlinks can be characterized as

$$\mathcal{D}^{\leftrightarrow} = \{(i, \mathcal{J}) :\ \{i\} \cup \mathcal{J} = \{h\} \cup \mathcal{K} \text{ for some } (h, \mathcal{K}) \in \mathcal{D}\}. \tag{16}$$

2.3 DECOMPOSABLE HYPERGRAPHS

In this section we introduce decomposable hypergraphs and describe some of their properties. The main goal of this section is to prove Lemma 2.3.10, a fundamental theoretic result that will be applied in Chapter 6 to the study of correlated equilibria of separable games.

In this section we only consider simple hypergraphs $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, i.e., hypergraphs such that if $\mathcal{J}, \mathcal{K} \in \mathcal{L}$ and $\mathcal{J} \supset \mathcal{K} \in \mathcal{L}$, then $\mathcal{J} = \mathcal{K}$. We start by giving some definitions.

Definition 2.3.1 (Hypergraph decomposition). *Given an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, a decomposition of $\mathcal{H}$ is a couple of sets $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$ such that $\mathcal{A} \cup \mathcal{B} = \mathcal{V}$, $\mathcal{A} \cap \mathcal{B} \subset \mathcal{J}$ for some $\mathcal{J} \in \mathcal{L}$, and for each hyperlink $\mathcal{K} \in \mathcal{L}$ either $\mathcal{K} \subset \mathcal{A}$ or $\mathcal{K} \subset \mathcal{B}$. A decomposition is said to be proper if both $\mathcal{A}, \mathcal{B} \neq \mathcal{V}$.*

Definition 2.3.2 (Decomposable hypergraphs). *An hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is said to be decomposable if either $\mathcal{V} \in \mathcal{L}$ or there exists a proper decomposition $\mathcal{A}, \mathcal{B}$ of $\mathcal{H}$ such that both the induced hypergraphs $\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}$ are decomposable.*

Note that the recursive definition of decomposable hypergraph is well founded since at each step of the recursion both hypergraphs $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ have fewer nodes than $\mathcal{H}$.

Remark 2.3.3. *Definition 2.3.2 is equivalent to that given in [64, Section 2.2.2]. Indeed, by iteratively applying Definition 2.3.2 to a decomposable hypergraph $\mathcal{H}$, one can decompose it into completely connected hypergraphs. It remains to be proven that at each step the H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is the direct join $\wedge$ (see [64, page 22]) of the induced H-graphs $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$. To see this observe that $\mathcal{H}_{\mathcal{A}} \wedge \mathcal{H}_{\mathcal{B}}$ is completely connected, as we have that $\mathcal{A} \cap \mathcal{B} \subset \mathcal{J}$ for some $\mathcal{J} \in \mathcal{L}$. Moreover $\mathcal{H}_{\mathcal{A}} \vee \mathcal{H}_{\mathcal{B}} = \mathcal{H}$. Indeed for any $\mathcal{J} \in \mathcal{L}$ we have that either $\mathcal{J} \subset \mathcal{A}$ or $\mathcal{J} \subset \mathcal{B}$. In the first case $\mathcal{J} = \mathcal{J} \cap \mathcal{A} \in \mathcal{L}_{\mathcal{A}}$ so that $\mathcal{J}$ is an hyperlink of $\mathcal{H}_{\mathcal{A}} \vee \mathcal{H}_{\mathcal{B}}$. The second case is analogous. On the other hand, for any hyperlink $\mathcal{J}$ of $\mathcal{H}_{\mathcal{A}} \vee \mathcal{H}_{\mathcal{B}}$ either $\mathcal{J} \in \mathcal{L}_{\mathcal{A}}$ or $\mathcal{J} \in \mathcal{L}_{\mathcal{B}}$. In the first case, $\mathcal{J} = \mathcal{K} \cap \mathcal{A} \subset \mathcal{K}$ for some $\mathcal{K} \in \mathcal{L}$. The second case is analogous.*

We now give a characterization of hypergraph decomposability in terms of the following property, known as the running intersection property.

Definition 2.3.4 (Running intersection property). *A sequence $(\mathcal{B}_k)_{k=1}^K$ of distinct subsets of $\mathcal{V}$ satisfies the running intersection property if for all $1 < i \leq K$ there exists $1 < j < i$ such that $\mathcal{S}_i \subset \mathcal{B}_j$ where*

$$\mathcal{S}_i = \mathcal{B}_i \cap (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_{i-1}). \tag{17}$$

The set of separators $\mathcal{S}$ is $\mathcal{S} = \{\mathcal{S}_j : 1 < j \leq K\}$ and for each separator $\mathcal{T} \in \mathcal{S}$ its combinatorial index is

$$\nu(\mathcal{T}) = \frac{1}{2} |\{j : 1 < j \leq K, \mathcal{S}_j = \mathcal{T}\}|$$

Proposition 2.3.5. *An hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is decomposable if and only if its hyperlinks can be ordered to form a sequence satisfying the running intersection property.*

Proof. This follows directly from [64, Propositions 2.27–2.29]. □

Based on the above characterization, we can link decomposability of hypergraphs to the corresponding property of graphs, given in Definition 2.2.2.

Definition 2.3.6 (Simplicial set). *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a set $\mathcal{B} \subset \mathcal{V}$ is simplicial if it has complete boundary, i.e., if the set $bd(\mathcal{B}) = \{j : (i, j) \in \mathcal{E}$ for some $i \in \mathcal{B}\} \setminus \mathcal{B}$ is complete in $\mathcal{G}$.*

Definition 2.3.7 (Perfect sequence). *A perfect sequence for a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a sequence $(\mathcal{B}_k)_{k=1}^{K}$ of complete sets of nodes of $\mathcal{G}$ such that, defining the history $\mathcal{I}_j$ and the residue $\mathcal{R}_j$ as in*

$$\mathcal{I}_j = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_j, \qquad \mathcal{R}_j = \mathcal{B}_j \setminus \mathcal{I}_{j-1}, \tag{18}$$

it holds that $\mathcal{R}_j$ is simplicial in $\mathcal{G}_{\mathcal{I}_j}$.

Proposition 2.3.8. *Consider a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$. $\mathcal{H}$ is decomposable if and only if $\mathcal{H} = \mathcal{H}_{\mathcal{C}\ell}(\mathcal{G})$ for some triangulated graph $\mathcal{G}$. In this case, any ordering $(\mathcal{B}_k)_{k=1}^{|\mathcal{L}|}$ of the hyperlinks of $\mathcal{H}$ that satisfies the running intersection property is a perfect sequence of $\mathcal{G}$. Finally, the set $\mathcal{S}$ of separators and the combinatorial index $\nu(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{S}$ are the same for every ordering of the hyperlinks that satisfies the running intersection property.*

Proof. The first statement follows directly from [64, Theorem 2.25] and [64, Proposition 2.5]. Notice that each $\mathcal{B}_j$ is complete, since it is an hyperlink of the clique hypergraph of $\mathcal{G}$.

By the relation among $\mathcal{G}$ and $\mathcal{H}$, we have that $\mathrm{bd}(\mathcal{R}_j) = \bigcup\{\mathcal{J} \in \mathcal{L} : \mathcal{J} \cap \mathcal{R}_j \neq \emptyset\} \setminus \mathcal{R}_j$ and its completeness in $\mathcal{G}_{\mathcal{I}_j}$ can be restated as $\mathrm{bd}(\mathcal{R}_j) \cap \mathcal{I}_j \subset \mathcal{K}$ for some $\mathcal{K} \in \mathcal{L}$, or equivalently

$$\bigcup\{\mathcal{J} \in \mathcal{L} : \mathcal{J} \cap \mathcal{R}_j \neq \emptyset\} \cap \mathcal{I}_j \subset \mathcal{R}_j \cup \mathcal{K}.$$

We prove that this holds for $\mathcal{K} = \mathcal{B}_j$ by showing that for all $1 \le j \le |\mathcal{L}|$ and for all $1 \le k \le |\mathcal{L}|$ either $\mathcal{B}_k \cap \mathcal{R}_j = \emptyset$ or $\mathcal{B}_k \cap \mathcal{I}_j \subset \mathcal{B}_j$. The statement is trivial for $k = j$ or if $k < j$, by the definition of $\mathcal{R}_j$. It remains to be proven that for all $j < k \le |\mathcal{L}|$ it holds $\mathcal{B}_k \cap \mathcal{R}_j \neq 0 \Rightarrow \mathcal{B}_k \cap \mathcal{I}_j \subset \mathcal{B}_j$. We do this by induction on k. If $k = j + 1$ then $\mathcal{B}_k \cap \mathcal{I}_j = S_k$ and by the running intersection property there exists $i < k$ such that $S_k \subset \mathcal{B}_i$. If $i = j$ we are done, as $\mathcal{B}_k \cap \mathcal{I}_j \subset \mathcal{B}_j$. Otherwise, if $i < j$ then $\mathcal{B}_k \cap \mathcal{R}_j = \emptyset$, as $\mathcal{B}_k \cap \mathcal{R}_j \subset (\mathcal{B}_k \cap \mathcal{I}_j) \cap \mathcal{R}_j \subset \mathcal{B}_i \cap \mathcal{R}_j = \emptyset$. This proves the base-case. We then assume the statement is true for $j+1, \ldots, k-1$ and prove it for k. By the running intersection property, there exists $i < k$ such that $S_k \subset \mathcal{B}_i$. If $i \le j$, $\mathcal{B}_k \cap \mathcal{I}_j \subset S_k \subset \mathcal{B}_i$ and we can proceed as in the base case. If instead $i > j$ we can apply the inductive hypothesis to i and we obtain that $\mathcal{B}_i \cap \mathcal{R}_j = \emptyset$ or $\mathcal{B}_i \cap \mathcal{I}_j \subset \mathcal{B}_j$. In the first case it follows that $\mathcal{B}_k \cap \mathcal{R}_j = \emptyset$ as $\mathcal{B}_k \cap \mathcal{R}_j \subset S_k \cap \mathcal{R}_j \subset \mathcal{B}_i \cap \mathcal{R}_j = \emptyset$. If instead $\mathcal{B}_i \cap \mathcal{I}_j \subset \mathcal{B}_j$ then also $\mathcal{B}_k \cap \mathcal{I}_j \subset \mathcal{B}_j$ since $\mathcal{B}_k \cap \mathcal{I}_j \subset S_k \cap \mathcal{I}_j \subset \mathcal{B}_i \cap \mathcal{I}_j \subset \mathcal{B}_j$. This proves the statement.

Finally, by the previous point, considering any two orderings $\mathcal{B} = (\mathcal{B}_k)_{k=1}^{|\mathcal{L}|}$ and $\mathcal{C} = (\mathcal{C}_k)_{k=1}^{|\mathcal{L}|}$ of $\mathcal{L}$, both are also perfect sequences of $\mathcal{G}$. As stated in [64, page 1278], this entails that $\mathcal{B}$ and $\mathcal{C}$ have the same set of separators $\mathcal{S}$, where each separator $\mathcal{T}$ is repeated the same number of times, namely $\nu(\mathcal{T})$. □

We can finally give the main statement of this section, concerning probabilistic models over hypergraphs. The following Lemma 2.3.10 identifies decomposability as a key structural property that allows to build a probability distribution over an hypergraph with prescribed hyperlink marginals, provided that they are consistent in the following sense.

Definition 2.3.9 (Consistent distributions). *Given two subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$ two distributions $P_{\mathcal{A}} \in \Delta(\mathcal{X}_{\mathcal{A}})$ and $P_{\mathcal{B}} \in \Delta(\mathcal{X}_{\mathcal{B}})$ are consistent if their marginals on $\mathcal{A} \cap \mathcal{B}$ coincide, namely $P_{\mathcal{A}}(x_{\mathcal{A}\cap\mathcal{B}}) = P_{\mathcal{B}}(x_{\mathcal{A}\cap\mathcal{B}})$ for all $x_{\mathcal{A}\cap\mathcal{B}} \in \mathcal{X}_{\mathcal{A}\cap\mathcal{B}}$.*

The next result will be crucial in Chapter 6.

Lemma 2.3.10. *Consider a decomposable hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ and a family of consistent distributions $P_{\mathcal{J}} \in \Delta(\mathcal{X}_{\mathcal{J}})$ for $\mathcal{J} \in \mathcal{L}$. The unique distribution that can be factorized over $\mathcal{H}$ and having the given distributions as its hyperlink marginals is given by*

$$Q(x) = \frac{\prod_{\mathcal{J}\in\mathcal{L}} P_{\mathcal{J}}(x_{\mathcal{J}})}{\prod_{\mathcal{T}\in\mathcal{S}} P_{\mathcal{T}}(x_{\mathcal{T}})^{\nu(\mathcal{T})}} \tag{19}$$

where for every separator $\mathcal{T}$ the distribution $P_{\mathcal{T}}$ is the marginal of $P_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{L}$ such that $\mathcal{T} \subset \mathcal{J}$.

Proof. This is a consequence of [39, Theorem 2.6] and Proposition 2.3.8. □

2.4 GAME-THEORETIC PRELIMINARIES

Throughout the thesis we shall consider strategic form games with finite nonempty player set $\mathcal{V}$ and a finite nonempty action set $\mathcal{A}_i$ for each player i in $\mathcal{V}$. We shall denote by $\mathcal{X} = \prod_{i\in\mathcal{V}} \mathcal{A}_i$ the space of all players' strategy profiles, also called configurations, and, for $\mathcal{J} \subset \mathcal{V}$, with $\mathcal{X}_{\mathcal{J}} = \prod_{i\in\mathcal{J}} \mathcal{A}_i$ the space of strategy profiles of players in $\mathcal{J}$. To ease the notation, for every player i in $\mathcal{V}$ we let $\mathcal{X}_{-i} = \prod_{j\in\mathcal{V}\setminus\{i\}} \mathcal{A}_j$ to be the set of strategy profiles of all players except for player i. As customary, for a strategy profile x in $\mathcal{X}$, the strategy profile of all players except for i is denoted by x_{-i} in $\mathcal{X}_{-i}$, while $x_{\mathcal{J}}$ denotes the strategy profiles restricted to a subset $\mathcal{J} \subseteq \mathcal{V}$.

We shall refer to two strategy profiles x and y in $\mathcal{X}$ as i-comparable and write $x \sim_i y$ when $x_{-i} = y_{-i}$, i.e., when x and y coincide except for possibly their i-th entries.

We let each player i in $\mathcal{V}$ be equipped with an utility function $u_i : \mathcal{X} \to \mathbb{R}$. We shall identify a game with player set $\mathcal{V}$ and strategy profile space $\mathcal{X}$ with the vector u assembling all the players' utilities. Notice that, in this way, the set of all games with player set $\mathcal{V}$ and strategy profile space $\mathcal{X}$, to be denoted as $\mathcal{U}$, is isomorphic to the vector space $\mathbb{R}^{\mathcal{V}\times\mathcal{X}}$. In view of this, we consider the standard euclidean scalar product on $\mathcal{U} = \mathbb{R}^{\mathcal{V}\times\mathcal{X}}$.

In strategic games, agents are assumed to be rational, i.e., they choose their action with the aim of improving or maximizing their utility. A strategy profile y in $\mathcal{X}$ is said to be a better response to $x \in \mathcal{X}$ for a player $i \in \mathcal{V}$ if $y \sim_i x$ and $u_i(y) \geq u_i(x)$. If the previous inequality holds strictly, then y is said to be a strict better response to x for player i. In particular, y is said to be a best response to x for player i if $y \in \arg\max_{z\sim_i x} u_i(z)$. Notice that every best response y to a configuration x for a player i is also a better response to x, but not vice versa. A (strict) better response path of length ℓ is a sequence of distinct strategy profiles $(x(t))_{t=0}^{\ell}$ such that each couple of consecutive strategies $(x(t), x(t+1))$ for $t \in \{0, \dots, \ell-1\}$ is comparable, i.e. there is some player $i \in \mathcal{V}$, referred to as the t-th active player, such that $x(t) \sim_i x(t+1)$, and $x(t+1)$ is a (strict) better response to $x(t)$ for such player i.

Given the actions of others, the *best response* (BR) correspondence returns the set of the best actions for agent i to play, that is, the actions that achieve the highest utility

$$\mathcal{B}_i(x_{-i}) = \underset{x_i\in\mathcal{A}_i}{\arg\max}\, u_i(x_i, x_{-i})\,.$$

A *pure strategy Nash equilibrium* (or simply Nash equilibrium for short) is a configuration x^* in $\mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*)\,, \qquad \forall i \in \mathcal{V}\,.$$

According to this, in a Nash equilibrium no player has any incentive in unilaterally deviating from their current action. By slightly weakening this condition, we may allow players to possibly have a small incentive in unilateral deviation, measured by a positive scalar ϵ, by introducing the following approximate notion of equilibrium. An ϵ-Nash equilibrium is a configuration x^* in $\mathcal{X}$ such that

$$u_i(x_i^*) \geq u_i(y_i, x_{-i}^*) - \epsilon\,, \qquad \forall i \in \mathcal{V}, \forall y_i \in \mathcal{A}_i\,.$$

Nash equilibria are a standard and intuitive solution concept for games [72]. However, it is well known that not every finite game admits a pure strategy Nash equilibrium, in particular the following is a classical example of 2-player binary action game for which the set of Nash equilibria is empty.

Example 2.4.1 (2-player discoordination game)**.** *Consider the* 2*-player binary actions game* u *described by the following utility function*

$$u_1(x_1, x_2) = \begin{cases} +1 & \text{if } x_1 = x_2 \\ -1 & \text{if } x_1 \neq x_2 \end{cases}\,, \quad u_2(x_1, x_2) = -u_1(x_1, x_2) \tag{20}$$

where $x_1, x_2 \in \{0, 1\}$. *This is known as the discoordination game or matching pennies game, since it describes the situation where one player aims at coordinating while the other wants to play the opposite action to that of their opponent. The discoordination game admits no Nash equilibria.*

This motivates the introduction of extended notions of equilibrium for which existence is always guaranteed. An instance that we will consider in our analysis is the following. *Correlated equilibria* (CE) [12] are a rich and widespread solution concept for games. The interest in this notion is justified by both modelling and computational considerations. Indeed, CE arise as the result of assuming the players to be Bayesian rational, which is often the case in economic or more general game theoretic settings [13]. Moreover, in contrast to Nash equilibria, CE can be computed efficiently for games in normal form [12].

Definition 2.4.2 (Correlated equilibria). *A correlated equilibrium (CE) of a game* u *is a probability distribution* P *over the game's configuration space* $\mathcal{X}$ *such that*

$$\forall i \in \mathcal{V}, \forall a, b \in \mathcal{A}_i, \quad \mathbb{E}_{x \sim P|x_i = a} \left[u_i(a, x_{-i}) \right] \geq \mathbb{E}_{x \sim P|x_i = a} \left[u_i(b, x_{-i}) \right]. \tag{21}$$

In the following, we denote by $CE(u)$ the set of correlated equilibria of the game u.

A game u is referred to as *non-strategic* if the utility of each player i in $\mathcal{V}$ does not depend on their own action, i.e., if

$$u_i(x) = u_i(y), \qquad \forall x, y \in \mathcal{X} \text{ s.t. } y \sim_i x. \tag{22}$$

The set of non-strategic games will be denoted by $\mathcal{N}$. Two games u and $\tilde{u}$ are referred to as *strategically equivalent* if their difference is a non-strategic game, namely $u - \tilde{u} \in \mathcal{N}$, .i.e., if

$$u_i(x) - \tilde{u}_i(x) = u_i(y) - \tilde{u}_i(y), \qquad \forall x, y \in \mathcal{X} \text{ s.t. } y \sim_i x. \tag{23}$$

Strategic equivalence is in fact an equivalence relation on the space of games $\mathcal{U}$ and we shall denote the strategic equivalence class of a game u by $[u]$. In the following we will often focus on properties of a game that are invariant with respect to strategic equivalence.

Notice that, as for Nash equilibria, correlated equilibria are preserved by strategic equivalence, as shown by the following.

Proposition 2.4.3. *For any two strategically equivalent games* u, v *in* $\mathcal{U}$, $CE(u) = CE(v)$.

Proof. Consider a correlated equilibrium $P \in CE(u)$ and a player $i \in \mathcal{V}$. By the strategic equivalence assumption, we can write

$$v_i(x) = u_i(x) + n_i(x_{-i}), \quad \forall x \in \mathcal{X}.$$

It then follows that for any two actions $a, b \in \mathcal{A}_i$

$$\begin{aligned}
\mathbb{E}_{x\sim P|x_i=a} v_i(a, x_{-i}) &= \mathbb{E}_{x\sim P|x_i=a} u_i(a, x_{-i}) + n_i(x_{-i}) \\
&= \mathbb{E}_{x\sim P|x_i=a} u_i(a, x_{-i}) + \mathbb{E}_{x\sim P|x_i=a} n_i(x_{-i}) \\
&\geq \mathbb{E}_{x\sim P|x_i=a} u_i(b, x_{-i}) + \mathbb{E}_{x\sim P|x_i=a} n_i(x_{-i}) \\
&= \mathbb{E}_{x\sim P|x_i=a} u_i(b, x_{-i}) + n_i(x_{-i}) \\
&= \mathbb{E}_{x\sim P|x_i=a} v_i(b, x_{-i}),
\end{aligned}$$

so that $P \in CE(v)$. By the arbitrariness of P, we conclude that $CE(u) \subset CE(v)$. The other inclusion is obtained analogously. □

This means that all games belonging to the same strategic equivalence class share the same Nash and correlated equilibria.

A game u is referred to as *normalized* if

$$\sum_{y\sim_i x} u_i(y) = 0\,, \qquad \forall x \in \mathcal{X}, i \in \mathcal{V}\,. \tag{24}$$

Notice that normalized games coincide with $\mathcal{N}^\perp$, i.e., they are the orthogonal space to non-strategic games. The *normalized version* of a game u is the game $\overline{u} \in \mathcal{U}$ with utilities

$$\overline{u}_i(x) = u_i(x) - \frac{1}{|\mathcal{A}_i|} \sum_{y\sim_i x} u_i(y)\,, \qquad \forall x \in \mathcal{X}, i \in \mathcal{V}\,. \tag{25}$$

It is then easily verified that the game $\overline{u}$ is both normalized and strategically equivalent to u. In fact, $\overline{u}$ is the unique normalized game in the strategic equivalence class $[u]$ [31, Lemma 4.6.].

A class of games that play a key role in the theory are potential games [69]. A game u in $\mathcal{U}$ is referred to as an (exact) *potential game* if there exists a *potential function* $\phi : \mathcal{X} \to \mathbb{R}$ such that

$$u_i(x) - u_i(y) = \phi(x) - \phi(y)\,, \tag{26}$$

for every player i in $\mathcal{V}$ and every pair of i-comparable strategy profiles $x \sim_i y$ in $\mathcal{X}$. Observe that (26) may be rewritten as $u_i(x) - \phi(x) = u_i(y) - \phi(y)$ for every $x \sim_i y$ in $\mathcal{X}$, which is in turn equivalent to that $u_i(x) - \phi(x) = n_i(x_{-i})$ does not depend on x_i for every strategy profile x in $\mathcal{X}$. Hence, u in $\mathcal{U}$ is a potential game with potential ϕ if and only if it is strategically equivalent to a game u^{ϕ} in $\mathcal{U}$ with utilities $u_i^{\phi}(x) = \phi(x)$ for every player i in $\mathcal{V}$. Potential games are particularly relevant in that the potential property of a game guarantees the existence of at least one Nash equilibrium, which is a maximum point of the potential function.

Example 2.4.4. *Consider a game with three players $\mathcal{V} = \{1, 2, 3\}$, binary action spaces $\mathcal{A}_i = \{-1, 1\}$ for $i = 1, 2, 3$, and utility functions*

$$u_1(x) = x_1x_2\,, \qquad u_2(x) = x_2(x_1 - x_3)\,, \qquad u_3(x) = -x_3x_2\,. \tag{27}$$

It is then straightforward to verify that this is a normalized potential game with potential function $\phi(x) = x_1x_2 - x_2x_3$.

Complementary to the class of potential games is that of harmonic games. A game u in $\mathcal{U}$ is referred to as *harmonic* [31] if

$$\sum_{i\in\mathcal{V}} \sum_{y\sim_i x} [u_i(x) - u_i(y)] = 0 \tag{28}$$

for every strategy profile x in $\mathcal{X}$. Notice that a normalized game is harmonic if and only if

$$\sum_{i\in\mathcal{V}} |\mathcal{A}_i| u_i(x) = 0\,, \qquad \forall x \in \mathcal{X}\,. \tag{29}$$

Hence, in particular, if the action sets of all players have the same cardinality $|\mathcal{A}_i| = a$, then a normalized game u is harmonic if and only if it is a *zero-sum game*, i.e.,

$$\sum_{i\in\mathcal{V}} u_i(x) = 0\,, \qquad \forall x \in \mathcal{X}\,. \tag{30}$$

Example 2.4.5. *Consider a game with player set and actions spaces as in Example 2.4.4 and utility functions*

$$u_1(x) = -x_1x_2x_3\,, \qquad u_2(x) = x_2x_3(x_1 - 1)\,, \qquad u_3(x) = x_3x_2\,. \tag{31}$$

It is then easily verified that this game is a zero-sum normalized game, hence a harmonic game since all action sets have the same cardinality.

Example 2.4.6 (Anticoordination and discoordination games.)**.** *We introduce the 2-player binary anticoordination game, with utility functions*

$$u_1(x_1, x_2) = \begin{cases} -1 & \text{if } x_1 = x_2 \\ +1 & \text{if } x_1 \neq x_2 \end{cases}\,, \qquad u_2(x_1, x_2) = u_1(x_1, x_2) \tag{32}$$

where $x_1, x_2 \in \{0,1\}$. The anticoordination game is a potential game. In contrast, the discordination game introduced in Example 2.4.1 does not admit any potential, as it does not have Nash equilibria. It can be verified that it is an harmonic game. Notice that both games are normalized.

We introduce the notations P and H to denote potential and harmonic games respectively.

2.4.1 *Graphical games*

Graphical games [59] are defined, with respect to a fixed graph $\mathcal{G}$, imposing that the utility of each player i only depends on the actions of players in their closed neighbourhood. We slightly depart from this and we assume that this holds up to non-strategic parts. This allows us a much more compact and clear presentation of our results. A similar point of view has been already considered in the literature [14]. The formal definition is the following.

Definition 2.4.7. *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a game u is said to be* graphical *with respect to $\mathcal{G}$, or to be a $\mathcal{G}$-game, if the utility of each player i in $\mathcal{V}$ can be decomposed as*

$$u_i(x) = v_i(x_i, x_{\mathcal{N}_i}) + n_i(x_{-i}), \tag{33}$$

where $v_i : \mathcal{X}_{\mathcal{N}_i^\bullet} \to \mathbb{R}$ is a function that depends on the action of player i and of players in the subset $\mathcal{N}_i$ only, while $n_i : \mathcal{X}_{-i} \to \mathbb{R}$ is a non-strategic component that does not depend on the action of player i.

By definition, the notion of graphicality introduced above is invariant with respect to strategic equivalence.

Notice that if a game u is graphical with respect to two graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$, it is also graphical with respect to their intersection $\mathcal{G}_1 \sqcap \mathcal{G}_2$. Since every game is trivially graphical on the complete graph on $\mathcal{V}$, we can conclude that to each game u in $\mathcal{U}$ one can always associate the smallest graph on which u is graphical. We shall refer to such graph as the *minimal graph of the game u* and denote it as $\mathcal{G}_u$.

In fact, important classes of graphical games allow for finer decompositions. In particular, for a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a *pairwise-separable network game* (cf. [34, 29]) on $\mathcal{G}$ is such that the utility of player i in $\mathcal{V}$ can be decomposed in the form

$$u_i(x) = \sum_{j \in \mathcal{N}_i} u_{ij}(x_i, x_j) \qquad \forall x \in \mathcal{X}, \tag{34}$$

where $u_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathbb{R}$ for i, j in $\mathcal{E}$. These games are also known as graphical polymatrix games on $\mathcal{G}$ [86] and are a special case of $\mathcal{G}$-games. In particular, when

$\mathcal{G}$ is undirected such games can be interpreted as follows. Players are identified with nodes of the graph and each pair of players $\{i, j\}$ connected by a link is involved in a two-player game having utility functions $u_{ij}(x_i, x_j)$ and $u_{ji}(x_j, x_i)$. Each player $i \in \mathcal{V}$ can choose an unique action $x_i \in \mathcal{A}_i$ to be used in all games they simultaneously participate in and they get a utility that is the linear aggregate of utilities from their outgoing links.

In the next chapter we will study a general notion of separability of games for which pairwise-separable network games are a special case. We end this section with two examples.

Example 2.4.8 (Network coordination game)**.** *For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a* network coordination game *on $\mathcal{G}$ is a game u where every player i in $\mathcal{V}$ has binary action set $\mathcal{A}_i = \{0, 1\}$ and utility function*

$$u_i(x) = \sum_{j \in \mathcal{N}_i} \zeta(x_i, x_j)\,, \tag{35}$$

where $\zeta(x_i, x_j) = \zeta(x_j, x_i)$ is a symmetric function such that $\zeta(0, 0) \geq \zeta(0, 1) = \zeta(1, 0)$ and $\zeta(1, 1) \geq \zeta(0, 1) = \zeta(1, 0)$. Clearly, every network coordination game with utilities (35) is a pairwise-separable game on $\mathcal{G}$. Moreover, if the graph $\mathcal{G}$ is undirected, a network coordination game on $\mathcal{G}$ is a potential game with potential function

$$\phi(x) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \zeta(x_i, x_j)\,.$$

In the special case when

$$\zeta(x_i, x_j) = \begin{cases} 1 & \text{if } x_i = x_j \\ -1 & \text{if } x_i \neq x_j \end{cases}$$

we obtain the well known majority game, which is used to model the behaviour of a population of conformist agents whose social interactions are described by $\mathcal{G}$.

2.4.2 *Flow representation of games*

In this section we consider games from a more abstract perspective, by following and extending the treatment of [31]. We recall that two strategies x and y in $\mathcal{X}$ are said to be comparable, denoted as $x \sim y$, if they coincide except for possibly the action of one player, say i, in which case we write $x \sim_i y$ and we say that x and y are i-comparable. Note that $\sim$ defines a symmetric and reflexive relation over $\mathcal{X}$. We distinguish the case when two distinct configurations x and y differ

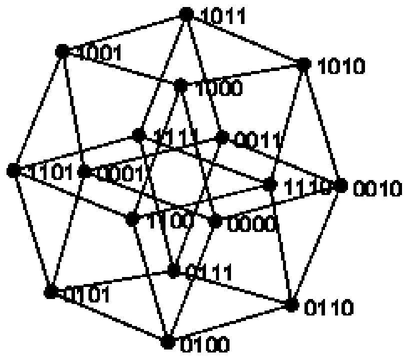


Figure 6: Configuration graph for a 4-player game with binary actions, denoted as 0 and 1.

exactly for the action of a single player by writing $x \sim_{\neq} y$. Note that, equivalently, two distinct configurations are comparable if and only if their Hamming distance d_H, defined as the number of components at which the two configurations are different, is exactly one. Formally, for x and y in $\mathcal{X}$ we have that

$$x \sim_{\neq} y \quad \iff \quad d_H(x, y) = 1.$$

This defines a symmetric relation $\mathcal{X}^{(2)}$ over the set of strategies $\mathcal{X}$:

$$\mathcal{X}^{(2)} = \left\{(x, y) \in \mathcal{X} \times \mathcal{X} \,|\, x \sim_{\neq} y\right\}.$$

Given a couple $(x, y) \in \mathcal{X}^{(2)}$, there exists exactly one player i such that $x_i \neq y_i$ and $x \sim_i y$. We introduce

$$F\ell = \left\{F \in \mathbb{R}^{\mathcal{X}^{(2)}} \,|\, F(x, y) = -F(y, x),\ \forall (x, y) \in \mathcal{X}^{(2)}\right\}$$

that is the set of alternating real valued functions defined over $\mathcal{X}^{(2)}$, which we call flows.

We can define the configuration graph $\mathcal{G}_{\text{conf}} = (\mathcal{X}, \mathcal{X}^{(2)})$, whose nodes are configurations and whose edges correspond to comparable configurations. $\mathcal{G}_{\text{conf}}$ is the graph product of $n = |\mathcal{V}|$ cliques: $\mathcal{G}_{\text{conf}} = \otimes_{i \in \mathcal{V}} \mathcal{C}_i(h_i)$, where $\mathcal{C}_i(h_i)$ is an h_i-clique associated to player i and $h_i = |\mathcal{A}_i|$ is the number of actions available to player i. See Figure 6 for a representation of the configuration graph of a game with 4 players and binary actions.

Notice that flows are defined on edges of $\mathcal{G}_{\text{conf}}$. For this reason we will often refer to elements of $\mathcal{X}^{(2)}$ as edges.

We can compute integrals of flows along paths on $\mathcal{G}_{\text{conf}}$ in the following sense. A path of length ℓ in $\mathcal{G}_{\text{conf}} = (\mathcal{X}, \mathcal{X}^{(2)})$ is a $(\ell + 1)$-tuple $\gamma = (\gamma_0, \ldots, \gamma_\ell)$ of strategies such that consecutive strategies are distinct and comparable, i.e. $(\gamma_i, \gamma_{i+1}) \in \mathcal{X}^{(2)}$, $i = 0, \dots, \ell - 1$. Then, for any flow $F \in F\ell$ and any path γ on $\mathcal{G}_{\text{conf}}$ we define the integral of F over γ as

$$\int_\gamma F := \sum_{i=0}^{\ell-1} F(\gamma_i, \gamma_{i+1}) \,.$$

We endow $F\ell$ with a scalar product, defined as

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{(x,y)\in\mathcal{X}^{(2)}} F_1(x,y) F_2(x,y).$$

The integral of a flow F over a cycle γ is also called *circulation*.

2.4.3 *Decomposition of games*

In this section we show how the potential-harmonic-nonstrategic decomposition of games, presented in [31], can be derived from the flow representation of games. The exposition is based on the work of [31], which we have adapted to our setting of finite games, thus deriving a simpler proof of the decomposition result that does not rely on Hodge theory.

We will come back to the potential-harmonic decomposition of games in Section 7.3, where we will analyse it from the point of view of separability. The understanding of the game decomposition and of its flow interpretation developed in these sections will allow to compute the properties of each game component explicitly to support the theoretical discussion with practical examples. An implementation of this procedure is reported in the following repository: `https://github.com/laura-arditti/game-decomposition`.

We start by defining some operators that act on the space of flows introduced in the previous section.

Definition 2.4.9. *We define the combinatorial gradient operator* $\mathrm{d} : \mathbb{R}^{\mathcal{X}} \to F\ell$ *whose action on the observable* $f \in \mathbb{R}^{\mathcal{X}}$ *gives the flow*

$$\mathrm{d}(f)(x,y) = f(y) - f(x) \,, \quad \forall (x,y) \in \mathcal{X}^{(2)} \,.$$

In Figure 7 we illustrate the action of the gradient operator.

It can be easily verified that the adjoint of the gradient d is the divergence operator $\mathrm{d}^* : F\ell \to \mathbb{R}^{\mathcal{X}}$ which acts on the flow F as

$$\mathrm{d}^*(F)(x) = - \sum_{y:(x,y)\in\mathcal{X}^{(2)}} F(x,y) \,, \quad \forall x \in \mathcal{X} \,.$$

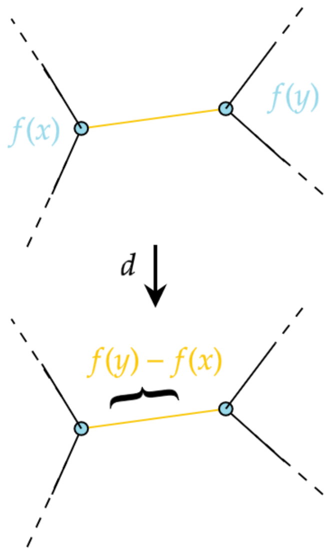


Figure 7: Representation of the action of the gradient operator: d maps the observabile f into a flow.

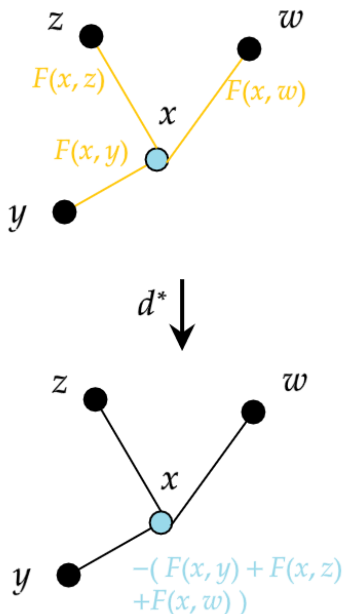


Figure 8: Representation of the action of the divergence operator: d^* gives the total inflow at each node of $\mathcal{G}_{\text{conf}}$.

i.e. giving the total inflow of F at x (see Figure 8).

Flows in the image $\mathrm{im}(\mathrm{d})$ of the gradient operator are called globally consistent, as their circulation over all cycles in $\mathcal{G}_{\text{conf}}$ is vanishing. It is not difficult to show that global consistency of a flow F is equivalent to F having vanishing circulation on all 3-cycles and 4-cycles of $\mathcal{G}_{\text{conf}}$.

We now define the operator D which creates a connection between games and flows.

Definition 2.4.10. *The operator $D : \mathcal{U} \to F\ell$ maps the game $u \in \mathcal{U}$ to the flow $Du = F \in F\ell$ such that*

$$F(x, y) = u_i(y) - u_i(x) \quad \forall (x, y) \in \mathcal{X}^{(2)}$$

where i is the only player such that x and y are i-comparable.

Example 2.4.11. *Consider a graphical game with 4 players and binary actions $\{0, 1\}$ over the graph $\mathcal{G}$ represented on the left in Figure 9. The three players $\{1, 2, 4\}$ (the blue nodes in Figure 9) are playing a coordination game (see Example 2.4.8) while the last player 3 (the red node) is playing an anti-coordination game (see Example 2.4.6) with each of their neighbors, resulting in the following utility functions*

$$u_i(x) = \delta_i \sum_{j \in \mathcal{N}_i} \zeta(x_i, x_j)$$

where ζ is as in Example 2.4.8 while $\delta_i = 1$ for coordinating players $\{1, 2, 4\}$ and $\delta_i = -1$ for the anti-coordinating player 3. Figure 9 (right) represents the resulting flow $F = Du$ over the configuration graph of the game, which is an hypercube.

The image of the operator D is the space $F\ell(\mathcal{U}) = \mathrm{im}\, D$ containing all flows which are generated by games. We will refer to flows in $F\ell(\mathcal{U})$ as *game flows*. We can characterize game flows with the following proposition.

Proposition 2.4.12. *The space of game flows coincides with the space of flows having vanishing circulation over all 3-cycles of $\mathcal{G}_{conf}$.*

Proof. Let $F \in F\ell(\mathcal{U})$ be a game flow and consider a 3-cycle γ of $\mathcal{G}_{\text{conf}}$ having the three strategies (x, y, z) as vertices. Notice that necessarily $x \sim_i y \sim_i z$ for some player $i \in \mathcal{V}$. We have

$$\begin{aligned}\int_\gamma F &= F(x, y) + F(y, z) + F(z, x) \\ &= u_i(y) - u_i(x) + u_i(z) - u_i(y) + u_i(x) - u_i(z) = 0\end{aligned}$$

We now show that $\int_\gamma F = 0$, $\forall \gamma$ 3-cycle in $\mathcal{G}_{\text{conf}}$ implies that $F \in F\ell(\mathcal{U})$. Given F with vanishing circulation over all 3-cycles in $\mathcal{G}_{\text{conf}}$, we construct the utility functions u_i, $i \in \mathcal{V}$, as follows. For each player i, we choose arbitrarily an action $a_i \in \mathcal{A}_i$

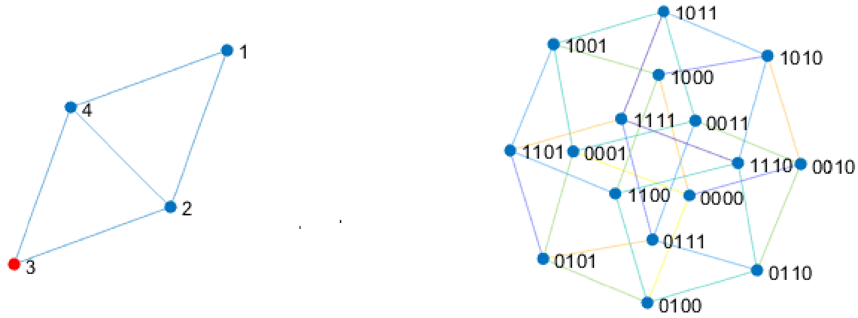


Figure 9: Graph of the game considered in Example 2.4.11 (left) and representation of the game flow over the configuration graph (right). Brightness of the configuration graph's edges correspond to intensity of the corresponding flow component.

and we fix arbitrarily the values of their utility $u_i(a_i, x_{-i})$ for each configuration $x_{-i} \in \mathcal{A}_{-i}$ of the other players. Then, the values of u_i on the remaining strategy profiles $x \in \mathcal{X}$ can be obtained as:

$$\forall x \in \mathcal{X}, \quad u_i(x_i, x_{-i}) = u(a_i, x_{-i}) + F\left((a_i, x_{-i}), (x_i, x_{-i})\right)$$

In this way the utilities u_i are well defined and $F = Du$, so that $F \in F\ell(\mathcal{U})$. $\square$

The kernel of the operator D can also be interpreted in terms of games: it is easy to see that it coincides with the space of non-strategic games, i.e. $\ker D = \mathcal{N}$. As a consequence, for a game $u \in \mathcal{U}$ we have that $[u] = D^{-1}Du$, i.e. the class of games strategically equivalent to u is the counter image under D of the flow produced by u.

Moreover, potential games can be characterized as the games u such that the corresponding flow Du lies in $\operatorname{im} \mathrm{d}$. More precisely, we have that $\operatorname{im} \mathrm{d} = DP$, i.e. potential games are the one producing globally consistent flows. To see this, it is sufficient to observe that for $\phi \in \mathbb{R}^{\mathcal{X}}$, $\mathrm{d}\,\phi = Du^{\phi}$ where u^{ϕ} is a potential game where each player i has utility $u_i^{\phi} = \phi$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \phi \in \mathbb{R}^{\mathcal{X}} & \xrightarrow{\;d\;} & F \in F\ell(\mathcal{U}) \\ \big\downarrow & \nearrow_{D} & \\ u^{\phi} \in \mathcal{U} & & \end{array}$$

Conversely, given a potential game $u \in P$ with potential ϕ, u is strategically equivalent to u_ϕ and $Du = Du_\phi = \mathrm{d}\,\phi$.

Since all game flows have vanishing circulation on 3-cycles of $\mathcal{G}_{\text{conf}}$, the previous result also implies that potential games can be characterized as games whose flow has vanishing circulation on 4-cycles of $\mathcal{G}_{\text{conf}}$.

Since $\operatorname{im}\mathrm{d} \subset F\ell(\mathcal{U})$, we can obtain an orthogonal decomposition of game flows with respect to the flows of potential games:

$$\begin{aligned} F\ell(\mathcal{U}) &= \operatorname{im}\mathrm{d} \perp \Big((\operatorname{im}\mathrm{d})^\perp \cap F\ell(\mathcal{U})\Big) \\ &= \operatorname{im}\mathrm{d} \perp (\ker \mathrm{d}^* \cap F\ell(\mathcal{U})) \\ &= DP \perp DH \end{aligned}$$

where $\perp$ denotes the orthogonal sum. In view of this result the space of harmonic games, which we introduced in (28), can be equivalently defined as follows.

Definition 2.4.13. *$u \in \mathcal{U}$ is an harmonic game if the related flow $F = Du$ has vanishing divergence, namely*

$$\mathrm{d}^*(F) \equiv 0\,.$$

In the following we will write $F = F_P + F_H$ to denote the orthogonal decomposition of $F \in F\ell(\mathcal{U})$. D is an onto operator in $F\ell(\mathcal{U})$. Due to this fact, we can pull back the decomposition of game flows to the space of games, as illustrated in Figure 10.

Proposition 2.4.14. *The space of game can be decomposed as*

$$\mathcal{U} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$$

where $\oplus$ denotes the direct sum. $\mathcal{P}$ is the space of normalized potential games, $\mathcal{N}$ is the space of non-strategic games, $\mathcal{H}$ is the space of normalized harmonic games.

Proof. From the decomposition of game flows, by just computing the counter image via operator D, we can obtain a decomposition of games as $\mathcal{U} = D^{-1}(DP) + D^{-1}(DH) = P + H$, which is not direct. Since the decomposition $F\ell(\mathcal{U}) = \operatorname{im} D = DP \perp DH$ is orthogonal, we have that $P \cap H = \ker D = \mathcal{N}$. Then we can write

$$\begin{aligned} \mathcal{U} &= \mathcal{N} \perp (\mathcal{N}^\perp \cap (P + H)) \\ &= \mathcal{N} \perp ((\mathcal{N}^\perp \cap P) \oplus (\mathcal{N}^\perp \cap H)) \\ &= \mathcal{N} \perp (\mathcal{P} \oplus \mathcal{H}) \end{aligned}$$

where by definition we have that $\mathcal{P} = \mathcal{N}^\perp \cap P$ are normalized potential games and $\mathcal{H} = \mathcal{N}^\perp \cap H$ are normalized harmonic games. Note that only the first sum is orthogonal while all sums are direct. $\square$

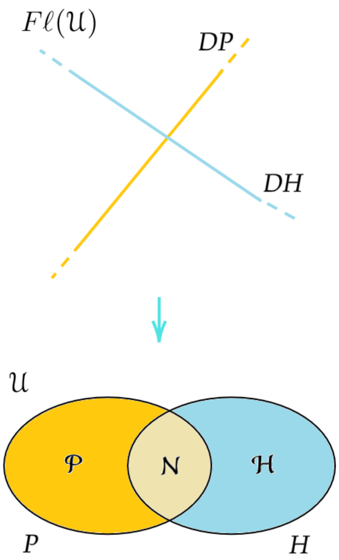


Figure 10: Direct sum decomposition of the space of games, obtained by pulling back the orthogonal decomposition of game flows.

Example 2.4.15. *Consider a game with three players $\mathcal{V} = \{1, 2, 3\}$, binary action spaces $\mathcal{A}_i = \{-1, 1\}$ for $i = 1, 2, 3$, and utility functions*

$$\begin{aligned} u_1(x) &= x_1 x_2 + (1 - x_1) x_2 x_3 \,, \\ u_2(x) &= x_2(x_1 - 2x_3) + (x_2 + 1) x_1 x_3 \,, \\ u_3(x) &= x_1 x_2 \,. \end{aligned} \tag{36}$$

Notice that the game can be decomposed as $u = u_{\mathcal{N}} + u_{\mathcal{P}} + u_{\mathcal{H}}$, where $u_{\mathcal{N}}$ is a is non-strategic game with utilities

$$u_{\mathcal{N},1}(x) = x_2 x_3 \,, \qquad u_{\mathcal{N},2}(x) = x_1 x_3 \,, \qquad u_{\mathcal{N},3}(x) = x_1 x_2 \,, \tag{37}$$

$u_{\mathcal{P}}$ is the normalized potential game with utilities as in (27), and $u_{\mathcal{H}}$ is the normalized harmonic game with utilities as in (31).

2.5 MARKOV RANDOM FIELDS

Here we present some facts about Markov Random Fields (MRF), which will play a role in the theory of separable potential games and, in particular, in the discussion of Section 7.2. Our presentation follows the book [64], to which we refer for more details on the topic.

For a finite set $\mathcal{V}$ we will consider random fields $(X_i)_{i \in \mathcal{V}}$, which are collections of random variables taking values in state spaces $(\mathcal{A}_i)_{i \in \mathcal{V}}$, which we assume to be discrete sets. For a subset $\mathcal{A} \subset \mathcal{V}$, we will let $\mathcal{X}_{\mathcal{A}} = \times_{i \in \mathcal{A}} \mathcal{A}_i$ and $\mathcal{X} = \mathcal{X}_{\mathcal{V}}$. We shall refer to the random vector X as positive if its probability distribution is equivalent to the product of the marginals, namely, if $\mathbb{P}(X = x) > 0$ whenever $\mathbb{P}(X_i = x_i) > 0$ for every i in $\mathcal{V}$.

Conditional independence will play a key role, as Markov properties, the core concept at the base of MRF, are the expression of conditional independence statements concerning a random field. For discrete random variables, to which we restrict in this discussion, the definition of conditional independence simplifies to the following. Given a random field $(X_i)_{i \in \mathcal{V}}$ and two sets $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$, we say that $X_{\mathcal{A}}$ is independent on $X_{\mathcal{B}}$, given (or conditional on) $X_{\mathcal{C}}$, and we write $X_{\mathcal{A}} \perp X_{\mathcal{B}} | X_{\mathcal{C}}$, if

$$\mathbb{P}(X_{\mathcal{A}} = x_a, X_{\mathcal{B}} = x_b \mid X_{\mathcal{C}} = x_c) = \mathbb{P}(X_{\mathcal{A}} = x_a \mid X_{\mathcal{C}} = x_c)\mathbb{P}(X_{\mathcal{B}} = x_b \mid X_{\mathcal{C}} = x_c).$$

To lighten the notation, we will write $\mathcal{A} \perp \mathcal{B} | \mathcal{C}$ in place of $X_{\mathcal{A}} \perp X_{\mathcal{B}} | X_{\mathcal{C}}$.

Markov random fields are *graphical models*, as they are characterized by properties that possess a graphical structure. Such structure can be described by assuming $\mathcal{V}$ to be the vertex set of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The main relevant cases for our purposes is when the graph $\mathcal{G}$ is an undirected graph. We then proceed by presenting Markov properties, which define Markov random fields.

2.5.1 *Markov properties*

As anticipated, Markov properties express conditional independence statements about a random field whose structure is described by an undirected graph. In particular, consider a random field $(X_i)_{i \in \mathcal{V}}$. Its probability measure $\mathbb{P}$ on $\mathcal{X}$ is said to obey, relative to the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,

- the pairwise Markov property (P) if for any pair (i, j) of non-adjacent vertices
$$i \perp j \mid \mathcal{V} \setminus \{i, j\},$$
- the local Markov property (L) if $\forall i \in \mathcal{V}$
$$i \perp \mathcal{V} \setminus \mathcal{N}_i^\bullet \mid \mathcal{N}_i,$$
- the global Markov property (G) if for any triple $(\mathcal{A}, \mathcal{B}, \mathcal{S})$ of disjoint subsets of $\mathcal{V}$ s.t. $\mathcal{S}$ separates $\mathcal{A}$ from $\mathcal{B}$
$$\mathcal{A} \perp \mathcal{B} \mid \mathcal{S},$$
where $\mathcal{S}$ separates $\mathcal{A}$ and $\mathcal{B}$ if all paths γ connecting two nodes $i \in \mathcal{A}$ and $j \in \mathcal{B}$ include nodes of $\mathcal{S}$.

There is a hierarchy of Markov properties, as for any undirected graph $\mathcal{G}$ and any probability measure $\mathbb{P}$ on $\mathcal{X}$,

$$(G) \Rightarrow (L) \Rightarrow (P),$$

meaning that the global property is the strongest, while the pairwise property is the weakest among Markov properties.

As conditional independence is closely related to factorization, so are Markov properties.

Definition 2.5.1 (Factorization property (F)). $\mathbb{P}$ *is said to factorize according to* $\mathcal{G}$ *if for all maximal cliques* $\mathcal{C} \subset \mathcal{V}$ *there exist non-negative functions* $k^{\mathcal{C}}$ *and a product measure* $\mu = \times_{i \in \mathcal{V}} \mu_i$ *on* $\mathcal{X}$ *s.t.* $\mathbb{P}$ *has density* f *with respect to* μ *where*

$$f(x) = \prod_{\mathcal{C} \in \mathcal{C}\ell(\mathcal{G})} k^{\mathcal{C}}(x_{\mathcal{C}})$$

Factorization property is stronger than all Markov properties, as for any undirected $\mathcal{G}$ and any $\mathbb{P}$ on $\mathcal{X}$,

$$(F) \Rightarrow (G) \Rightarrow (L) \Rightarrow (P).$$

However, under positivity and continuity assumptions, the weakest Markov property implies factorization so that all the aforementioned properties are equivalent. This is established by the Hammersley-Clifford Theorem, a fundamental result in the theory or MRF.

Theorem 2.5.2 (Hammersley-Clifford). *For an undirected graph* $\mathcal{G}$ *and a distribution* $\mathbb{P}$ *with positive and continuous density* f *with respect to a product measure* μ

$$(F) \Leftrightarrow (P).$$

3

SEPARABILITY OF GAMES

In a graphical game, the way the utility of a player depends on the actions played by their neighbor players is a key feature that plays a crucial role in the analysis of the game (e.g., Nash equilibria, existence of a potential). In classical coordination (see Example 2.4.8) or anti-coordination games such dependence can be seen as the sum of pairwise interactions with each single neighbor player. In other cases, as in the best-shot public good games (Example 3.2.2), instead such decomposition is not possible.

While this finer structure of utility functions cannot be addressed within the notion of graphical game, it is at the core of the theory of separable games, that is the main focus of this thesis. Separable game is a more refined notion than graphical game and allows for a finer description of the dependence pattern among the players in a game.

In this section, we introduce the notions of *separable function* with respect to a H-graph and *separable game* with respect to a FDH-graph.

Throughout the section, we shall assume to have fixed a finite player set $\mathcal{V}$, nonempty action sets $\mathcal{A}_i$ for every player i in $\mathcal{V}$, and let $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$. We recall that $\mathcal{U}$ stands for the set of games with player set $\mathcal{V}$ and strategy profile set $\mathcal{X}$.

3.1 SEPARABLE GAMES

First of all, we introduce the notion of separability of a game with respect to a forward directed hypergraph (FDH-graph). The proposed notion of separable game refines that of graphical game.

Definition 3.1.1. *Given a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, a game u in $\mathcal{U}$ is* $\mathcal{F}$-separable *if the utility of each player i in $\mathcal{V}$ can be decomposed as*

$$u_i(x) = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}) + n_i(x_{-i}), \tag{38}$$

where $u_i^{\mathcal{J}} : \mathcal{A}_i \times \mathcal{X}_j \to \mathbb{R}$ are functions that depend on the actions of player i and of players in the subset $\mathcal{J}$ of head nodes of hyperlink $(i, \mathcal{J})$ only, while $n_i : \mathcal{X}_{-i} \to \mathbb{R}$ is a non-strategic component that does not depend on the action of player i.

Definition 3.1.1 captures not only locality of the relative influences among players in the game, but also the fact that players may have separate interactions with different groups of other players. Up to a non-strategic component, this grouping of the player set is modeled as a FDH-graph with node set coinciding with the player set $\mathcal{V}$ and where each group jointly influencing player i corresponds to a directed hyperlink with tail node i.

Separability of a game can be equivalently expressed in terms of a corresponding property of the single utility functions. Indeed, we can introduce the following notion of separability for a function defined on product spaces.

Definition 3.1.2. *A function* $f : \mathcal{X} \to \mathbb{R}$ *is* $\mathcal{H}$-separable, *where* $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ *is a H-graph, if there exist functions* $f_{\mathcal{J}} : \mathcal{X}_{\mathcal{J}} \to \mathbb{R}$*, for* $\mathcal{J}$ *in* $\mathcal{L}$*, such that*

$$f(x) = \sum_{\mathcal{J} \in \mathcal{L}} f_{\mathcal{J}}(x_{\mathcal{J}}), \qquad \forall x \in \mathcal{X}. \tag{39}$$

Separability of a function $f : \mathcal{X} \to \mathbb{R}$ with respect to a H-graph $\mathcal{H}$ thus consists in decomposability of $f(x)$ as a sum of functions each depending exclusively on the variables $x_{\mathcal{J}}$ associated to an undirected hyperlink $\mathcal{J}$ of $\mathcal{H}$.

As anticipated, we can express separability of a game in terms of separability of its utility functions. Indeed, given an FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, consider, for each i in $\mathcal{V}$, the 'local' H-graph

$$\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i), \quad \mathcal{L}_i = \{\{i\} \cup \mathcal{J} : (i, \mathcal{J}) \in \mathcal{D}\} \cup \{\mathcal{V} \setminus \{i\}\}. \tag{40}$$

Notice that a game u in $\mathcal{U}$ is $\mathcal{F}$-separable in the sense of Definition 3.1.1 if and only if, for every i in $\mathcal{V}$, the utility function u_i is $\mathcal{H}_i$-separable in the sense of Definition 3.1.2.

We now make a few technical remarks on the definition of separable games. In particular, we compare the current notion of separability for games with an equivalent model introduced in [77] and we describe its connections to graphical games.

Remark 3.1.3. *The notion of separable game introduced by Definition 3.1.1 is both a generalization and a refinement of the notion of graphical game. Indeed, a direct consequence of Definitions 2.4.7 and 3.1.1 and of relations (10) and (11) is that, given a FDH-graph* $\mathcal{F}$*, every* $\mathcal{F}$*-separable game* u *is graphical with respect to the graph* $\mathcal{G}^{\mathcal{F}}$*. Similarly, if* u *is graphical with respect to* $\mathcal{G}$*, it is* $\mathcal{F}^{\mathcal{G}}$*-separable.*

Remark 3.1.4. *By definition, the notion of separability introduced above is invariant with respect to strategic equivalence, i.e., a game* u *is* $\mathcal{F}$*-separable if and only every game* $\tilde{u}$ *that is strategically equivalent to* u *is* $\mathcal{F}$*-separable. In that, Definition 3.1.1 differs from other notions proposed in the literature, see, e.g., that of "graphical multi-hypermatrix game"*

[77]. According to the previous remark, such invariance property is common to both the Definition 2.4.7 of graphical games we choose to adopt and the Definition 3.1.1 of separable game that we introduce. The reason for this choice will become clear in the following Chapter 4 and is related to the existence and characterization of minimal separable representations for games. For the moment we limit ourselves to pointing out the following fact. If a normalized games is $\mathcal{F}$*-separable, it can be represented as in (38) where* $n_i(x_{-i}) \equiv 0$ *for every player* i *in* $\mathcal{V}$*.*

Remark 3.1.5. *For undirected FDH-graphs, Definition (3.1.1) corresponds to the notion of hypergraphical game introduced in [80], apart from the strategic equivalence issue discussed in the previous Remark 3.1.4.*

3.2 EXAMPLES

We propose two examples, which describe the relation between graphicality and separability of games and illustrate the two opposite extreme cases for the separability property of a graphical game.

Example 3.2.1 (Coordination game)**.** *The network coordination game on a graph* $\mathcal{G}$ *presented in Example 2.4.8 is* $\mathcal{F}$*-separable where* $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ *is the FDH-graph such that*

$$\mathcal{D} = \{(i, \{j\}) \,,\, i \in \mathcal{V}, j \in \mathcal{N}_i\} \tag{41}$$

and $\mathcal{N}_i$ *is the out-neighborhood of player* i *in* $\mathcal{G}$*. This observation actually holds more in general. All pairwise-separable network games with respect to a graph* $\mathcal{G}$ *are* $\mathcal{F}$*-separable with respect to the FDH-graph* $\mathcal{F}$ *as in (41), which is obtained interpreting* $\mathcal{G}$ *as a FDH-graph. This justifies our terminology, showing that pairwise-separable games possess the finest separability property, where players are affected* independently *by each of their neighbors.*

We next analyse a particular graphical game that is not pairwise-separable, but rather separable with respect to a different FDH-graph.

Example 3.2.2 (Best-shot public good game)**.** *Consider a graph* $\mathcal{G}$ *and the game where every player* i *in* $\mathcal{V}$ *has binary action set* $\mathcal{A}_i = \{0, 1\}$ *and utility:*

$$u_i(x) = \begin{cases} 1 - c & \text{if} \quad x_i = 1 \\ 1 & \text{if} \quad x_i = 0 \text{ and } x_j = 1 \text{ for some } j \in \mathcal{N}_i \\ 0 & \text{if} \quad x_i = 0 \text{ and } x_j = 0 \text{ for every } j \in \mathcal{N}_i \,. \end{cases}$$

The game u *constructed in this way is an instance the so called "public good games" [52]. It models a more complex behaviour for the population* $\mathcal{V}$ *compared to simple coordination:*

players benefit form acquiring some good, represented by taking action 1 *and which is public in the sense that it can be lent from one player to another. Taking action* 1 *has a cost* c*, so players would prefer that one of their neighbors takes that action, but taking the action and paying the cost is still the best choice if no one of their neighbors does. The best-shot public good game is a graphical game on* $\mathcal{G}$ *but it is not pairwise-separable. In fact,* u *is separable with respect to the FDH-graph* $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ *where*

$$\mathcal{D} = \{(i, \mathcal{N}_i), \ i \in \mathcal{V}\} .$$

This is the coarsest separability property for a $\mathcal{G}$*-graphical game, and it expresses the fact that players are affected* jointly *by all their neighbors.*

4

MINIMAL REPRESENTATION OF GAMES

In this chapter we focus on the properties of the separable representation of games introduced in Chapter 3. Our discussion originates from the observation that the decomposition of utility functions of separable games as a combination of local contributions is typically not unique. Local utility terms can be reassembled in various ways leading to substantially different representations associated with different FDH-graphs. However, the set of hyperlinks needed to describe the structure of a game determines its parametric complexity in inferential procedures. Similarly, the efficiency of algorithms to find Nash equilibria, as discussed in Section 1.3.2, is strictly connected to the properties of the game's representation. This indicates that finding parsimonious representations of separable games is a crucial problem.

The main results of this chapter concern the existence of a minimal separable representation for any game, which encodes only the essential dependencies among players. This is relevant from a modelling perspective as it allows to focus on those interactions that really affect the strategic behaviour of players. From a computational perspective, the size of a game representation is a major issue and we show that the separable representation does not provide significant improvements over the normal form representation. For this purpose we introduce the notion of strict separability, that solves this problem by providing a more compact representation of games at the cost of including in the model some inessential interactions among players.

4.1 EXISTENCE OF MINIMAL REPRESENTATIONS

In Section 2.4.1 we defined the graph $\mathcal{G}_u$ of the game u as the minimal graph $\mathcal{G}$ such that u is a $\mathcal{G}$-game. It is less trivial to see that the concept of minimal FDH-graph of a game is well defined, as we will show in this section.

First, notice the following general fact about the decomposition of functions. Every function $f : \mathcal{X} \to \mathbb{R}$ is $\mathcal{H}$-separable with respect to the trivial H-graph $\mathcal{H} = (\mathcal{V}, \{\mathcal{V}\})$ having a unique undirected hyperlink consisting of all nodes. Moreover, given two H-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \preceq \mathcal{H}_2$, we have that if f is $\mathcal{H}_1$-separable, then it is also $\mathcal{H}_2$-separable. In particular, a function f is $\mathcal{H}$-separable if and only if it is $\overline{\mathcal{H}}$-separable, where we recall that $\overline{\mathcal{H}}$ is the simple H-graph associated to $\mathcal{H}$.

Then, the following fundamental technical result holds true, which will be instrumental to our future derivations.

Lemma 4.1.1. *Let a function $f : \mathcal{X} \to \mathbb{R}$ be both $\mathcal{H}_1$-separable and $\mathcal{H}_2$-separable for two H-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$. Then, f is also $\mathcal{H}$-separable, where $\mathcal{H} = \mathcal{H}_1 \sqcap \mathcal{H}_2$.*

Proof. Let Σ_f be the family of all H-graphs $\mathcal{H}$ such that f is $\mathcal{H}$-separable and, for $i = 1, 2$, let $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ in Σ_f be an H-graph such that f is $\mathcal{H}_i$-separable. We can then write

$$f(x) = \sum_{\mathcal{J} \in \mathcal{L}_1} g^{(1)}_{\mathcal{J}}(x_{\mathcal{J}}) = \sum_{\mathcal{K} \in \mathcal{L}_2} g^{(2)}_{\mathcal{K}}(x_{\mathcal{K}}) \,, \qquad \forall x \in \mathcal{X} \,. \tag{42}$$

Then, for every $\mathcal{J}$ in $\mathcal{L}_1$, we have that

$$g^{(1)}_{\mathcal{J}}(x_{\mathcal{J}}) = \sum_{\mathcal{K} \in \mathcal{L}_2} g^{(2)}_{\mathcal{K}}(x_{\mathcal{K}}) - \sum_{\mathcal{I} \in \mathcal{L}_1 \setminus \{\mathcal{J}\}} g^{(1)}_{\mathcal{I}}(x_{\mathcal{I}}) \,, \qquad \forall x \in \mathcal{X} \,. \tag{43}$$

Now, observe that, since the lefthand side of (43) is independent from $x_{\mathcal{V} \setminus \mathcal{J}}$, so is its righthand side. Therefore, we may rewrite (43) as

$$g^{(1)}_{\mathcal{J}}(x_{\mathcal{J}}) = \sum_{\mathcal{K} \in \mathcal{L}_2} h^{(2)}_{\mathcal{K} \cap \mathcal{J}}(x_{\mathcal{K} \cap \mathcal{J}}) - \sum_{\mathcal{I} \in \mathcal{L}_1 \setminus \{\mathcal{J}\}} h^{(1)}_{\mathcal{I} \cap \mathcal{J}}(x_{\mathcal{I} \cap \mathcal{J}}) \,, \qquad \forall x \in \mathcal{X} \,, \tag{44}$$

where, for an arbitrarily chosen y in $\mathcal{X}$,

$$h^{(i)}_{\mathcal{K} \cap \mathcal{J}}(x_{\mathcal{K} \cap \mathcal{J}}) = g^{(i)}_{\mathcal{K}}(x_{\mathcal{K} \cap \mathcal{J}}, y_{\mathcal{K} \setminus \mathcal{J}}) \,, \qquad \forall i = 1, 2 \,, \quad \forall \mathcal{K} \in \mathcal{L}_1 \cup \mathcal{L}_2 \,, \quad \forall x \in \mathcal{X} \,. \tag{45}$$

It then follows from (42), (44) and (45) that

$$f(x) = \sum_{\mathcal{J} \in \mathcal{L}_1} \sum_{\mathcal{K} \in \mathcal{L}_2} h^{(2)}_{\mathcal{K} \cap \mathcal{J}}(x_{\mathcal{K} \cap \mathcal{J}}) - \sum_{\mathcal{J} \in \mathcal{L}_1} \sum_{\mathcal{I} \in \mathcal{L}_1 \setminus \{\mathcal{J}\}} h^{(1)}_{\mathcal{I} \cap \mathcal{J}}(x_{\mathcal{I} \cap \mathcal{J}}) \,, \qquad \forall x \in \mathcal{X} \,. \tag{46}$$

Observe that (46) is not yet the desired separability decomposition because of the presence of the second term in its righthand side. However, a suitable iterative application of (46) allows us to prove the claim. To formally see this, it is convenient to first introduce the following definition. Given a H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, let the H-graphs $\sqcap^k \mathcal{H} = (\mathcal{V}, \mathcal{L})$ be defined by

$$\mathcal{L} = \{\mathcal{J}_1 \cap \cdots \cap \mathcal{J}_k \mid \mathcal{J}_s \in \mathcal{L} \; \forall s, \; \mathcal{J}_s \neq \mathcal{J}_t \; \forall s \neq t\} \tag{47}$$

and notice that $\sqcap^2(\sqcap^k \mathcal{H}) \preceq \sqcap^{k+1} \mathcal{H}$. We can now interpret (46) as saying that

$$(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^2 \mathcal{H}_1) \in \Sigma_f \,. \tag{48}$$

We now prove by induction that, for every $k \geq 2$,

$$(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1) \in \Sigma_f \,. \tag{49}$$

Indeed, assume that (49) holds true for a certain k and let us prove it for $k+1$. Considering that (48) is true for any pair of H-graphs $\mathcal{H}_1, \mathcal{H}_2$ in Σ_f, if we apply it replacing $\mathcal{H}_1$ with $(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)$, we obtain that

$$(((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)) \sqcap \mathcal{H}_2) \sqcup (\sqcap^2((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1))) \in \Sigma_f \,. \tag{50}$$

Notice now that

$$((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)) \sqcap \mathcal{H}_2 \preceq \mathcal{H}_1 \sqcap \mathcal{H}_2 \,, \tag{51}$$

$$\sqcap^2((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)) \preceq (\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^2(\sqcap^k \mathcal{H}_1)) \preceq (\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^{k+1} \mathcal{H}_1) \,. \tag{52}$$

Relations (50), (51), and (52) imply (49) for $k+1$. Therefore, (49) holds true for every value of k. Finally, notice that, for $k > |\mathcal{L}|$, $\sqcap^k \mathcal{H}$ is the H-graph with an empty set of hyperlinks. This proves that $\mathcal{H}_1 \sqcap \mathcal{H}_2 \in \Sigma_f$. □

Lemma 4.1.1 and the foregoing considerations motivate the following definition.

Definition 4.1.2. *A H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is a* minimal H-graph *for a function $f : \mathcal{X} \to \mathbb{R}$ if $\mathcal{H}$ is simple, f is $\mathcal{H}$-separable, and $\mathcal{H} \preceq \tilde{\mathcal{H}}$ for every H-graph $\tilde{\mathcal{H}} = (\mathcal{V}, \tilde{\mathcal{L}})$ such that f is $\tilde{\mathcal{H}}$-separable.*

We can now prove the following result.

Proposition 4.1.3. *Every function $f : \mathcal{X} \to \mathbb{R}$ admits a unique minimal H-graph $\mathcal{H}_f$.*

Proof. Lemma 4.1.1 implies that f is $\mathcal{H}$-separable, where $\mathcal{H}$ is the $\sqcap$-intersection of all H-graphs $\tilde{\mathcal{H}} = (\mathcal{V}, \tilde{\mathcal{L}})$ such that f is $\tilde{\mathcal{H}}$-separable. Then, f is also $\overline{\mathcal{H}}$-separable. Now, let $\mathcal{H}_f = \overline{\mathcal{H}}$ and notice that $\mathcal{H}_f \preceq \mathcal{H} \preceq \tilde{\mathcal{H}}$ for every H-graph $\tilde{\mathcal{H}}$ such that f is $\tilde{\mathcal{H}}$-separable. Since by construction $\mathcal{H}_f$ is simple, we have that $\mathcal{H}_f$ is the minimal H-graph of f. □

A game u can be $\mathcal{F}$-separable with respect to different FDH-graphs $\mathcal{F}$. In fact, if a game is $\mathcal{F}_1$-separable for a given FDH-graph $\mathcal{F}_1$, it is also $\mathcal{F}_2$-separable for every FDH-graph $\mathcal{F}_2$ such that $\mathcal{F}_1 \preceq \mathcal{F}_2$. Hence, in particular, a game u is $\mathcal{F}$-separable if and only if it is $\overline{\mathcal{F}}$-separable.

A natural question, addressed below, is whether there exists a FDH-graph $\mathcal{F}$ that captures the minimal structure of a game. Such minimality property is formalized by the following definition.

Definition 4.1.4. *A FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ is a* minimal FDH-graph *of a game u in $\mathcal{U}$ if $\mathcal{F}$ is simple, u is $\mathcal{F}$-separable, and $\mathcal{F} \preceq \tilde{\mathcal{F}}$ for every FDH-graph $\tilde{\mathcal{F}} = (\mathcal{V}, \tilde{\mathcal{D}})$ such that u is $\tilde{\mathcal{F}}$-separable.*

The following result states that every game admits a minimal FDH-graph with respect to which it is separable.

Theorem 4.1.5. *Every game u in $\mathcal{U}$ admits a unique minimal FDH-graph $\mathcal{F}_u = (\mathcal{V}, \mathcal{D})$.*

Proof. Let u in $\mathcal{U}$ be a game that is separable with respect to two FDH-graphs $\mathcal{F}_1 = (\mathcal{V}, \mathcal{D}_1)$ and $\mathcal{F}_2 = (\mathcal{V}, \mathcal{D}_2)$. For every player i in $\mathcal{V}$, consider the corresponding local H-graphs $\mathcal{H}_i^1$ and $\mathcal{H}_i^2$ as defined in (40). We have that the utility function u_i is $\mathcal{H}_i^s$-separable for $s = 1, 2$ and, consequently, because of Lemma 4.1.1, also $\mathcal{H}_i^1 \sqcap \mathcal{H}_i^2$-separable. Since, for i in $\mathcal{V}$, $\mathcal{H}_i^1 \sqcap \mathcal{H}_i^2$ are the local H-graphs associated with the intersection $\mathcal{F}_1 \sqcap \mathcal{F}_2$, we deduce that u is $\mathcal{F}_1 \sqcap \mathcal{F}_2$-separable. Then, the result follows arguing in the same way as in the proof of Proposition 4.1.3. □

Theorem 4.1.5 guarantees existence of minimal separable representations for games, but does not suggest a way to check for minimality of a given separable representation or to obtain the minimal representation for a given game. This topic will be addressed in a general setting in Chapter 8. Instead, the following examples show how to perform these tasks for specific games.

Example 4.1.6 (Best-shot public good game cont.). *Consider the best-shot public good game u with respect to a graph $\mathcal{G}$, as illustrated in Example 3.2.2. The minimal FDH-graph for u is $\mathcal{F}_u = (\mathcal{V}, \mathcal{D}_u)$ where*

$$\mathcal{D}_u = \{(i, \mathcal{N}_i),\ i \in \mathcal{V}\}\,. \tag{53}$$

To show this, take $i \in \mathcal{V}$ and suppose there exist two sets $\mathcal{J}, \mathcal{K} \subsetneq \mathcal{N}_i$ and two functions $u_i^{\mathcal{J}} : \mathcal{A}_i \times \mathcal{X}_{\mathcal{J}} \to \mathbb{R}$ and $u_i^{\mathcal{K}} : \mathcal{A}_i \times \mathcal{X}_{\mathcal{K}} \to \mathbb{R}$ such that u_i can be decomposed as

$$u_i(x) = u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}) + u_i^{\mathcal{K}}(x_i, x_{\mathcal{K}}). \tag{54}$$

Equation (54) entails that $\mathcal{J} \cup \mathcal{K} \supset \mathcal{N}_i$ so that there exist two players $j \in \mathcal{N}_i \setminus \mathcal{K}$ and $k \in \mathcal{N}_i \setminus \mathcal{J}$. Denote with $\delta_h \in \mathcal{X}$ for $h \in \mathcal{V}$ the game configuration where all players' actions are 0 except for player h, who plays 1. We obtain the following equations

$$\begin{cases} 0 = u_i(\mathbf{0}) = u_i^{\mathcal{J}}(\mathbf{0}) + u_i^{\mathcal{K}}(\mathbf{0}) \\ 1 = u_i(\delta_k) = u_i^{\mathcal{J}}(\mathbf{0}) + u_i^{\mathcal{K}}(\delta_k) \\ 1 = u_i(\delta_j) = u_i^{\mathcal{J}}(\delta_j) + u_i^{\mathcal{K}}(\mathbf{0}) \\ 1 = u_i(\delta_j + \delta_k) = u_i^{\mathcal{J}}(\delta_j) + u_i^{\mathcal{K}}(\delta_k). \end{cases}$$

Since these equations are incompatible it follows that (54) *cannot hold when both* $\mathcal{J}$ *and* $\mathcal{K}$ *are strict subsets of* $\mathcal{N}_i$*. As a result, the decomposition* (53) *is indeed minimal.*

Example 4.1.7 (Two-level coordination game)**.** *We consider the following variation of the network coordination game presented in Example 2.4.8. We fix a set of players* $\mathcal{V}$ *and the same action set for all players:* $\mathcal{A} = \mathcal{A}_i = \{0, 1\}$ *for all* i *in* $\mathcal{V}$*. For every pair of players* i, j*, we consider functions* $u_{ij} : \mathcal{A}^2 \to \mathbb{R}$ *defined as the pairwise utility function* ζ *of the network coordination game* (35)*. We now consider an undirected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and, for every* i *in* $\mathcal{V}$*, functions* $\tilde{u}_i : \mathcal{X}_{\mathcal{N}_i^\bullet} \to \mathbb{R}$ *given by*

$$\tilde{u}_i(x) = \begin{cases} L & \text{if } x_i = x_k \text{ for all } k \in \mathcal{N}_i \\ 0 & \text{otherwise,} \end{cases}$$

where $L > 0$*. We finally define the utility of player* i *as:*

$$u_i(x) = \sum_{j \neq i} u_{ij}(x_i, x_j) + \tilde{u}_i(x_{\mathcal{N}_i^\bullet})$$

The interpretation is the following: each player has a benefit that is in part linearly proportional to the number of individuals playing the same action and, additionally, they receive an extra value L *if the agent's action is in complete agreement with their neighbors'. This type of utility function models, for example, the situation where players' actions represent the acquisition of a new technology and the benefit to a player comes from two channels: the range of diffusion of the technology in the whole population and the opportunity to use such technology with their strict collaborators. If we consider the FDH-graph* $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ *where*

$$\mathcal{D} = \{(i, \{j\}), j \neq i\} \cup \{(i, \mathcal{N}_i), i \in \mathcal{V}\}$$

we have that the game u *is* $\mathcal{F}$*-separable. However, notice that this is not the minimal FDH-graph for* u *as the minimal one is* $\mathcal{F}_u = (\mathcal{V}, \mathcal{D}_u) = \overline{\mathcal{F}}$ *where*

$$\mathcal{D}_u = \{(i, \{j\}), j \notin \mathcal{N}_i^\bullet\} \cup \{(i, \mathcal{N}_i), i \in \mathcal{V}\}.$$

The minimal separability and graphicality properties of a game are connected. More specifically, the relation between the minimal FDH-graph $\mathcal{F}_u$ of a game u and its minimal graph $\mathcal{G}_u$ is clarified in the following result.

Corollary 4.1.8. *For every game* u *in* $\mathcal{U}$*, the minimal graph* $\mathcal{G}_u$ *and the minimal FDH-graph* $\mathcal{F}_u$ *are related by* $\mathcal{G}_u = \mathcal{G}^{\mathcal{F}_u}$*.*

Proof. We know from Remark 3.1.3 that u is a $\mathcal{F}^{\mathcal{G}_u}$-separable $\mathcal{G}^{\mathcal{F}_u}$-game. From this and the relations (12) we have that

$$\mathcal{G}_u \subseteq \mathcal{G}^{\mathcal{F}_u}, \quad \mathcal{F}_u \preceq \mathcal{F}^{\mathcal{G}_u} \Rightarrow \mathcal{G}^{\mathcal{F}_u} \subseteq \mathcal{G}^{(\mathcal{F}^{\mathcal{G}_u})} = \mathcal{G}_u.$$

This concludes the proof. □

4.2 STRICT SEPARABILITY

To study the properties of strategic equivalent games we need to give a more expressive definition of separability that is able to distinguish games in the same strategic equivalence class. This can be done based on the separability of the games' utility functions as follows.

Let u be a game and assume that for each $i \in \mathcal{V}$ the utility function u_i can be decomposed according to an hypergraph $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ as

$$u_i(x) = \sum_{\mathcal{J} \in \mathcal{L}_i} u_i^{\mathcal{J}}(x_{\mathcal{J}}). \tag{55}$$

Then, as previously observed, the separability of the utilities u_i corresponds to u being separable on the FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ such that

$$\mathcal{D} = \{(i, \mathcal{J} \setminus \{i\}) | i \in \mathcal{V}, \mathcal{J} \in \mathcal{L}_i\}$$

Notice that, in contrast with (40), here we do not require i to be contained in all $\mathcal{J} \in \mathcal{L}_i$. Separability of games can then be equivalently defined as follows.

Definition 4.2.1. *A game u in $\mathcal{U}$ is* separable *on a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, if there exist H-graphs $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ for each $i \in \mathcal{V}$ such that u_i is $\mathcal{H}_i$-separable and*

$$\mathcal{D} = \{(i, \mathcal{J} \setminus \{i\}) | i \in \mathcal{V}, \mathcal{J} \in \mathcal{L}_i, i \in \mathcal{J}\} \tag{56}$$

Similarly, we can give the following definition.

Definition 4.2.2 (Strict separability)**.** *A game u in $\mathcal{U}$ is* strictly separable *on a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, if there exist H-graphs $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ for each $i \in \mathcal{V}$ such that u_i is $\mathcal{H}_i$-separable and*

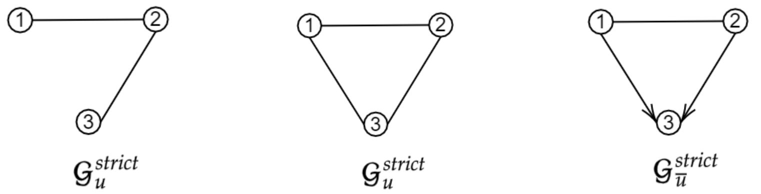
$$\mathcal{D} = \{(i, \mathcal{J} \setminus \{i\}) | i \in \mathcal{V}, \mathcal{J} \in \mathcal{L}_i\} \tag{57}$$

Strict separability of a game u on $\mathcal{F}$ implies that the utility of each player i in $\mathcal{V}$ can be decomposed as

$$u_i(x) = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}), \tag{58}$$

where $u_i^{\mathcal{J}} : \mathcal{A}_i \times \mathcal{X}_{\mathcal{J}} \to \mathbb{R}$ are functions that depend on the actions of player i and of players in the subset $\mathcal{J}$ of head nodes of hyperlink $(i, \mathcal{J})$ only.

Remark 4.2.3. *The proposed notion of strictly separable game is equivalent to that of graphical multi-hypermatrix game [77]. Referring to the terminology of [77, Definition 1], given a strictly $\mathcal{F}$-separable game u, the corresponding graphical multi-hypermatrix game has local cliques $\mathcal{C}_i = \{\{i\} \cup \mathcal{J}, (i, \mathcal{J}) \in \mathcal{D}\}$ and local-clique payoff matrices $M'_{i,C}(x_C) = u_i^{C \setminus \{i\}}(x_C)$ for $i \in \mathcal{V}$ and $C \in \mathcal{C}_i$.*



(a) Strict graph of the normalized potential game in Example 4.2.5.

(b) Strict graph of the game in Example 4.2.6.

(c) Strict graph of normalized version of the game in Example 4.2.6.

Figure 11: Strict graphs of the games for Examples 4.2.5–4.2.6

From the notion of strict separability we can derive a corresponding notion of strict graphicality.

Definition 4.2.4 (Strict graphicality). *A game $u \in \mathcal{U}$ is* strictly $\mathcal{G}$-graphical *if it is strictly $\mathcal{F}^{\mathcal{G}}$-separable.*

As for strict separability, strict graphicality of a game u allows to characterize the utility of each player i in $\mathcal{V}$ as

$$u_i(x) = v_i(x_i, x_{\mathcal{N}_i}), \tag{59}$$

where $v_i : \mathcal{A}_i \times \mathcal{X}_{\mathcal{N}_i} \to \mathbb{R}$ is a function that depends on the action of player i and of players in the subset $\mathcal{N}_i$ only and where, in contrast with (33), no non-strategic terms appear.

Again, if a game u is strictly graphical with respect to two graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$, it is also strictly graphical with respect to their intersection $\mathcal{G}_1 \sqcap \mathcal{G}_2$. Since every game is trivially strictly graphical on the complete graph on $\mathcal{V}$, we can conclude that to each game u in $\mathcal{U}$ one can always associate the smallest graph on which u is strictly graphical. We shall refer to such graph as the *minimal strict graph of the game* u and denote it as $\mathcal{G}_u^{\text{strict}}$.

The following examples illustrate how both undirected and directed graphs can naturally arise as the strict graphs of games. They also show how strict minimal graphs of two strategically equivalent games can be quite different.

Example 4.2.5. *Consider the normalized potential game of Example 2.4.4 with utility functions*

$$u_1(x) = x_1 x_2\,, \qquad u_2(x) = x_2(x_1 - x_3)\,, \qquad u_3(x) = -x_3 x_2\,. \tag{60}$$

For this game $\mathcal{G}_u^{strict}$ is an undirected graph with link set $\mathcal{E} = \{(1,2),(2,1),(2,3),(3,2)\}$ (see Figure 11a).

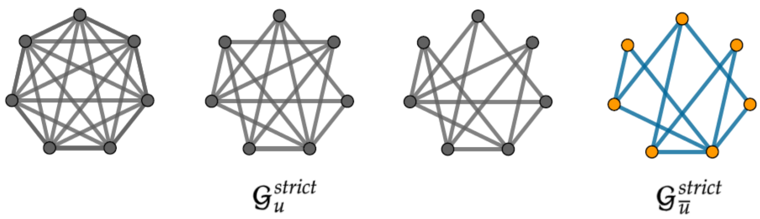


Figure 12: Representation of minimal strict-graphs for four strategically equivalent games with 7 players. On the left: a complete graph. On the right: the minimal strict-graph of the class.

Example 4.2.6. *Consider the game u of Example 2.4.15. In this case $\mathcal{G}^{strict}_u$ is the complete graph of order 3, i.e., it has link set $\mathcal{E} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$, as displayed in Figure 11b. The normalized version $\overline{u}$ of u with utilities (37) has minimal graph $\mathcal{G}^{strict}_{\overline{u}}$ with link set $\mathcal{E} = \{(1,2),(1,3),(2,1),(2,3)\}$, as displayed in Figure 11c. Notice that, albeit the normalized game $\overline{u}$ is strategically equivalent to u, in this case $\mathcal{G}^{strict}_{\overline{u}} \subsetneq \mathcal{G}^{strict}_u$ is a strict subgraph.*

As illustrated by Example 4.2.6, two strategically equivalent games u and $\tilde{u}$ might have quite different strict graphs $\mathcal{G}^{\text{strict}}_u$ and $\mathcal{G}^{\text{strict}}_{\tilde{u}}$. Indeed, it is easy to see that every game u admits a strategically equivalent game $\tilde{u}$ such that $\mathcal{G}^{\text{strict}}_{\tilde{u}}$ is the complete graph (see also Figure 12).

It is not obvious whether a strategical equivalence class $[u]$ always contains a game whose strict graph is minimal and whose strict FDH-graph is the finest in the class. This property turns out to hold true, as a consequence of the following result.

Theorem 4.2.7. *Let u be an $\mathcal{F}$-separable game. Then its normalized version $\overline{u}$ is strictly $\mathcal{F}$-separable. Moreover, denoting by $\mathcal{F}_u$ the minimal FDH-graph such that u is $\mathcal{F}_u$-separable, $\bar{u}$ is minimally strictly separable on $\mathcal{F}_u$.*

Proof. Let the $\mathcal{F}$-separable game u have utilities satisfying (38). Then, for $(i, \mathcal{J})$ in $\mathcal{D}$, the quantity

$$n_i^{\mathcal{J}}(x_{\mathcal{J}}) = \frac{1}{|\mathcal{A}_i|} \sum_{y_i \in \mathcal{A}_i} u_i^{\mathcal{J}}(y_i, x_{\mathcal{J}})$$

depends only on the actions of players in $\mathcal{J}$. Then, define a new game u^* with utilities

$$u_i^*(x) = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_i^{*\mathcal{J}}(x_i, x_{\mathcal{J}}) \,, \qquad u_i^{*\mathcal{J}}(x_i, x_{\mathcal{J}}) = u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}) - n_i^{\mathcal{J}}(x_{\mathcal{J}}) \,, \tag{61}$$

for every player i in $\mathcal{V}$ and strategy profile x in $\mathcal{X}$.

Notice that, by definition, u^* is strictly $\mathcal{F}$-separable. Moreover, since the terms $n_i^{\mathcal{J}}(x_{\mathcal{J}})$ do not depend on the action x_i of player i, the game u^* defined by (61) is strategically equivalent to u. For any $x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}$ it holds that

$$\begin{aligned}\sum_{y_i \in \mathcal{A}_i} u_i^{*\mathcal{J}}(y_i, x_{\mathcal{J}}) &= \sum_{y_i \in \mathcal{A}_i} \left(u_i^{\mathcal{J}}(y_i, x_{\mathcal{J}}) - n_i^{\mathcal{J}}(x_{\mathcal{J}}) \right) \\ &= \left(\sum_{y_i \in \mathcal{A}_i} u_i^{\mathcal{J}}(y_i, x_{\mathcal{J}}) \right) - |\mathcal{A}_i| n_i^{\mathcal{J}}(x_{\mathcal{J}}) = 0\,.\end{aligned}$$

Then, by linearity, u^* is normalized. But since there exists just one normalized game $\overline{u}$ in the strategic equivalence class $[u]$, this shows that $u^* = \overline{u}$. Finally, assume that $\mathcal{F}$ is minimal for u, i.e. $\mathcal{F} = \mathcal{F}_u$. To show that $\mathcal{F}$ is the minimal strict FDH-graph for $\bar{u}$, we call $\tilde{\mathcal{F}} = (\mathcal{V}, \tilde{\mathcal{D}})$ the minimal strict FDH-graph of $\bar{u}$ and we show that $\mathcal{F} \preceq \tilde{\mathcal{F}}$. Recall that since u is strategically equivalent to $\bar{u}$ we have

$$\begin{aligned}u_i(x) &= v_i(x_{-i}) + \bar{u}_i(x) \\ &= v_i(x_{-i}) + \sum_{(i,\mathcal{J}) \in \tilde{\mathcal{D}}} \bar{u}_{i,\mathcal{J}}(x_i, x_{\tilde{\mathcal{J}}})\end{aligned}$$

for some function v_i depending only on x_{-i}. The last shows that u is $\tilde{\mathcal{F}}$-separable. Since by assumption $\mathcal{F}$ is the minimal FDH-graph for u, we have that $\mathcal{F} \preceq \tilde{\mathcal{F}}$. □

4.3 REPRESENTATION COMPLEXITY

The two notions of separability of games given in Definition 3.1.1 and 4.2.2 have different and complementary purposes. In a modelling setting, separability (Definition 3.1.1) is more appropriate as it does not include non-strategic dependencies. As such, two players are involved in an hyperlink only if the actions of one affects the strategic behaviour of the other. As a consequence, the only dependencies that are explicitly represented are those that are common to all member of the strategic equivalence class of the game. In contrast, strict separability (Definition 4.2.2) might differ among member games of a common strategic equivalence class, which makes it less suitable from a modelling perspective. However, this representation is justified in a computational context. Indeed, separability of a game does not allow to bound its representation size, as the representation of the non-strategic part of the game, $(n_i)_{i \in \mathcal{V}}$ in (38), has by itself size comparable with the normal form representation of the game, i.e. $O(na^n)$ when $|\mathcal{A}_i| \leq a$, $\forall i \in \mathcal{V}$ and $|\mathcal{V}| = n$. On the other hand, any strictly separable game over $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ can be represented with size $O(|\mathcal{D}|a^j)$ whenever $|\mathcal{J}| \leq j$ $\forall (i, \mathcal{J}) \in \mathcal{D}$, which might be much smaller than the normal form representation. For the most part of this work, the focus is on structural properties of games which are better described in terms of their separable representation. However, when describing algorithms operating on separable games we will assume that the input is encoded in strict separable representation in order to evaluate the algorithms efficiency. Notice that these differences collapse when dealing with normalized games, for which separable and strict-separable representations coincide.

5

PROJECTIONS ONTO SEPARABLE GAMES

In the space $\mathcal{U} \simeq \mathbb{R}^{\mathcal{V}\times\mathcal{X}}$ of all games with set of players $\mathcal{V}$ and strategy profile space $\mathcal{X}$, we consider the classical (Euclidean) inner product

$$\langle u, v\rangle_{\mathcal{U}} = \sum_{i\in\mathcal{V}} \langle u_i, v_i\rangle_{\mathbb{R}^{\mathcal{X}}} = \sum_{i\in\mathcal{V}}\sum_{x\in\mathcal{X}} u_i(x)v_i(x). \tag{62}$$

This metrization turns out to be useful in the representation of separable games as well as in addressing the basic problem of measuring how far a given game u is from being separable with respect to a given FDH-graph and the related problem of computing the best (closest) approximation of it with some desired separability property. As separable games with respect to a given FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ form a linear subspace $\mathcal{U}_{\mathcal{F}}$ of $\mathcal{U}$, a basic point is to compute the orthogonal projection of a game u onto $\mathcal{U}_{\mathcal{F}}$. The main contribution of this chapter is an explicit formula for this. We notice that such approximation issues raise naturally in a number of contexts, for instance when a game, that is known to be separable with respect to a certain hypergraph structure, is learned from noisy data or when, instead, we want to enforce separability as an approximate modelling assumption.

5.1 EXPLICIT COMPUTATION OF PROJECTIONS

We start by introducing some technical definitions. Given $\mathcal{J} \subseteq \mathcal{V}$, we use the notation $\mathbb{R}^{\mathcal{X}}_{\mathcal{J}}$ and $\Pi_{\mathcal{J}}$ for, respectively, the subspace of functions in $\mathbb{R}^{\mathcal{X}}$ that only depend on the variables in $\mathcal{J}$ and the corresponding orthogonal projection. For an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, we let $\mathbb{R}^{\mathcal{X}}_{\mathcal{H}}$ to be the linear subspace of $\mathcal{H}$-separable functions. We denote by $\Pi_{\mathcal{H}}$ the orthogonal projector onto $\mathbb{R}^{\mathcal{X}}_{\mathcal{H}}$ with respect to the inner product $<,>_{\mathbb{R}^{\mathcal{X}}}$. Finally, let $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ be an FDH-graph. We denote with $\mathcal{U}_{\mathcal{F}}$ the linear subspace of $\mathcal{U}$ such that

$$\mathcal{U}_{\mathcal{F}} = \{u \in \mathcal{U} : u \text{ is } \mathcal{F}\text{-separable}\}.$$

Also, we define the orthogonal projection operator $\Pi_{\mathcal{F}} = \mathcal{U} \longrightarrow \mathcal{U}_{\mathcal{F}}$ such that any game $u \in \mathcal{U}$ can be uniquely decomposed as

$$u = u_{\mathcal{F}} + u_{\mathcal{F}^{\perp}} \tag{63}$$

where $u_{\mathcal{F}} = \Pi_{\mathcal{F}} u \in \mathcal{U}_{\mathcal{F}}$ and $u_{\mathcal{F}^{\perp}} = (I - \Pi_{\mathcal{F}})u \in (\mathcal{U}_{\mathcal{F}})^{\perp}$.

We are interested in characterizing $\Pi_{\mathcal{F}}$ and in providing a practical way to compute the projection $\Pi_{\mathcal{F}} u$.

We can now state the main result of this section, Theorem 5.1.1, which shows that the projection operates separately on each player and gives an explicit expression for its player-wise action.

Theorem 5.1.1. *For any game u and FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ the orthogonal projection $\Pi_{\mathcal{F}} : \mathcal{U} \longrightarrow \mathcal{U}_{\mathcal{F}}$ can be computed player-wise as follows*

$$(u_{\mathcal{F}})_i(x) = \sum_{\substack{\mathcal{S} \subset \mathcal{L}_i \\ \mathcal{S} \neq \emptyset}} (-1)^{|\mathcal{S}|+1} \frac{1}{|\mathcal{X}_{\mathcal{V} \setminus \cap \mathcal{S}}|} \sum_{y \in \mathcal{X}_{\mathcal{V} \setminus \cap \mathcal{S}}} u_i(x_{\cap \mathcal{S}}, y), \tag{64}$$

where we denoted $\cap \mathcal{S} = \cap_{\mathcal{J} \in \mathcal{S}} \mathcal{J}$ and $\mathcal{L}_i$ is the set of hyperlinks of the local H-graph $\mathcal{H}_i$ defined in (40).

Proof. For any set of players $\mathcal{J} \in \mathcal{V}$ we define the H-graph $\mathcal{H}_{\mathcal{J}} = (\mathcal{V}, \mathcal{L}_{\mathcal{J}}), \mathcal{L}_{\mathcal{J}} = \{\mathcal{J}\}$ and the linear operator $\Pi_{\mathcal{J}} : \mathbb{R}^{\mathcal{X}} \longrightarrow \mathbb{R}^{\mathcal{X}}_{\mathcal{J}}$ whose action, for all $f \in \mathbb{R}^{\mathcal{X}}$ and $x \in \mathcal{X}$, is given by

$$f_{\mathcal{J}}(x) = (\Pi_{\mathcal{J}} f)(x) = \frac{1}{|\mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}|} \sum_{y_{\mathcal{V} \setminus \mathcal{J}} \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}} f(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}). \tag{65}$$

In words, $\Pi_{\mathcal{J}}$ projects any function $f \in \mathbb{R}^{\mathcal{X}}$ to a function $f_{\mathcal{J}}$ that is separable on the H-graph $\mathcal{H}_{\mathcal{J}}$ containing a single hyperlink $\mathcal{J}$. We start by proving that $\Pi_{\mathcal{J}}$ is an orthogonal projection. Clearly $f_{\mathcal{J}} \in \mathbb{R}^{\mathcal{X}}_{\mathcal{J}}$ since, by taking the average in (65) over the actions of players in $\mathcal{V} \setminus \mathcal{J}$, the result depends only on the actions of players in $\mathcal{J}$, which is the only hyperlink in $\mathcal{H}_{\mathcal{J}}$. Moreover, it holds that $f - f_{\mathcal{J}} \in (\mathbb{R}^{\mathcal{X}}_{\mathcal{J}})^{\perp}$ since for each $g \in \mathbb{R}^{\mathcal{X}}_{\mathcal{J}}$ we have

$$\begin{aligned}
\langle f_{\mathcal{J}}, g \rangle &= \sum_{x \in \mathcal{X}} f_{\mathcal{J}}(x) g(x) \\
&= \sum_{x \in \mathcal{X}} \frac{1}{|\mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}|} \sum_{y_{\mathcal{V} \setminus \mathcal{J}} \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}} f(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) g(x) \\
&= \frac{1}{|\mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}|} \sum_{x \in \mathcal{X}} \sum_{y_{\mathcal{V} \setminus \mathcal{J}} \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}} f(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) g(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) \\
&= \sum_{x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}} \sum_{y_{\mathcal{V} \setminus \mathcal{J}} \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}} f(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) g(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) \sum_{x_{\mathcal{V} \setminus \mathcal{J}} \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}} \frac{1}{|\mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}|} \\
&= \sum_{x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}} \sum_{y_{\mathcal{V} \setminus \mathcal{J}} \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{J}}} f(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) g(x_{\mathcal{J}}, y_{\mathcal{V} \setminus \mathcal{J}}) \\
&= \langle f, g \rangle,
\end{aligned}$$

so that $\langle f - f_{\mathcal{J}}, g \rangle = 0$.

Based on $\Pi_{\mathcal{J}}$, the projection $\Pi_{\mathcal{H}} : \mathbb{R}^{\mathcal{X}} \longrightarrow \mathbb{R}^{\mathcal{X}}_{\mathcal{H}}$ onto the space of functions $\mathbb{R}^{\mathcal{X}}_{\mathcal{H}}$ that are separable with respect to an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ (with possibly multiple hyperlinks) can be computed by means of a mechanism of inclusion-exclusion:

$$f_{\mathcal{H}} = \Pi_{\mathcal{H}} f = \sum_{\substack{\mathcal{S} \subset \mathcal{L} \\ \mathcal{S} \neq \emptyset}} (-1)^{|\mathcal{S}|+1} \Pi_{\cap \mathcal{S}} f. \tag{66}$$

The previous formula, which is derived in Appendix 5.3, is a general fact on the orthogonal projection on a sum of subspaces (possibly with a non null intersection) and only depends on the fact that the individual projectors commute. It finally follows that for all $i \in \mathcal{V}$, $(\Pi_{\mathcal{F}})_i$ is an orthogonal projection onto $\mathbb{R}^{\mathcal{X}}_{\mathcal{H}_i}$ so that it can be expressed as

$$(\Pi_{\mathcal{F}} u)_i = \Pi_{\mathcal{H}_i} u_i = \sum_{\substack{\mathcal{S} \subset \mathcal{L}_i \\ \mathcal{S} \neq \emptyset}} (-1)^{|\mathcal{S}|+1} \Pi_{\cap \mathcal{S}} u_i.$$

□

The analysis performed in the general setting of separable games can be specialized to the case of graphical games, thus deriving a corollary of Theorem 5.1.1. Corollary 5.1.2 is interesting on its own, as graphical games form a relevant subfamily of separable games, but it is also interesting in that its very simple and intuitive formulation helps clarifying the meaning of Theorem 5.1.1.

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we consider the linear subspace of graphical games on $\mathcal{G}$, denoted by $\mathcal{U}_{\mathcal{G}}$, and, for any game u with player set $\mathcal{V}$, its orthogonal projection on $\mathcal{U}_{\mathcal{G}}$, which we denote by $u_{\mathcal{G}}$. We can write

$$u = \Pi_{\mathcal{G}} u + (I - \Pi_{\mathcal{G}}) u = u_{\mathcal{G}} + u_{\mathcal{G}^{\perp}} \tag{67}$$

where $\Pi_{\mathcal{G}}$ denotes the projection operator on $\mathcal{U}_{\mathcal{G}}$, $u_{\mathcal{G}} \in \mathcal{U}_{\mathcal{G}}$ and $u_{\mathcal{G}^{\perp}} \in \mathcal{U}_{\mathcal{G}}^{\perp}$.

If $\mathcal{G}$ is a graph over $\mathcal{V}$, we have observed (see Remark 3.1.3) that a game u is graphical with respect to $\mathcal{G}$ if and only if it is $\mathcal{F}^{\mathcal{G}}$-separable (see (10)). In this particular case the local H-graphs $\mathcal{H}_i$ only contain two hyperlinks, namely the closed neighbourhood $\mathcal{N}_i^{\bullet}$ and $\mathcal{V} \setminus \{i\}$. Consequently, the orthogonal projection $u_{\mathcal{G}}$ of a game u takes a simpler form that we explicitly characterize in the following result.

Corollary 5.1.2. *For any game $u \in \mathcal{U}$ and graph $\mathcal{G}$, the orthogonal projection $\Pi_{\mathcal{G}} : \mathcal{U} \to \mathcal{U}_{\mathcal{G}}$ can be computed player-wise as follows*

$$(u_{\mathcal{G}})_i(x) = \left(\Pi_{\mathcal{N}_i^\bullet} u_i\right)(x) = \frac{1}{|\mathcal{X}_{\mathcal{V}\setminus\mathcal{N}_i^\bullet}|} \sum_{y \in \mathcal{X}_{\mathcal{V}\setminus\mathcal{N}_i^\bullet}} u_i(x_{\mathcal{N}_i^\bullet}, y)$$
$$+ \frac{1}{|\mathcal{X}_i|} \sum_{y \in \mathcal{X}_i} u_i(y, x_{-i})$$
$$- \frac{1}{|\mathcal{X}_{\mathcal{V}\setminus\mathcal{N}_i}|} \sum_{y \in \mathcal{X}_{\mathcal{V}\setminus\mathcal{N}_i}} u_i(x_{\mathcal{N}_i}, y), \qquad \forall x \in \mathcal{X}\,. \tag{68}$$

where $\mathcal{N}_i$ and $\mathcal{N}_i^\bullet$ denote the open and closed out-neighbourhoods of player i in $\mathcal{G}$. Moreover, if u is normalized, then

$$(u_{\mathcal{G}})_i(x) = \left(\Pi_{\mathcal{N}_i^\bullet} u_i\right)(x) = \frac{1}{|\mathcal{X}_{\mathcal{V}\setminus\mathcal{N}_i^\bullet}|} \sum_{y \in \mathcal{X}_{\mathcal{V}\setminus\mathcal{N}_i^\bullet}} u_i(x_{\mathcal{N}_i^\bullet}, y)\,. \tag{69}$$

Corollary 5.1.2 shows that the projection (69) of a normalized game u on a graph $\mathcal{G}$ has a pretty intuitive interpretation: the utility $(u_{\mathcal{G}})_i$ of player $i \in \mathcal{V}$ in a configuration $x \in \mathcal{X}$ is just the average of all utility values of u_i over configurations that coincide with x over the neighborhood of player i in $\mathcal{G}$. The extra terms appearing in (68) only account for the possible presence of a non-strategic component in the more general case.

Remark 5.1.3. *Corollary 5.1.2 opens an interesting connection to the problem of learning graphical game models. In [41] the authors formulate an optimization problem to learn the graphical structure of a game from observation of the game's configurations and the corresponding utility outcomes. More precisely, they define a loss function that measures to what extent a graph structure is able to justify the game's observations and propose both exact and heuristic methods to solve the loss minimization problem, focusing on the performance of the proposed algorithms. In view of our results we can fit [41] into a formal framework. Indeed, it follows from Corollary 5.1.2 that if the set of game's observations is complete, the loss function proposed in [41] corresponds to the distance of the game from its orthogonal projection on a space of strict graphical games. The optimization problem then corresponds to finding the graph for which the projection is closest to the original game.*

5.2 APPROXIMATE NASH EQUILIBRIA

In this section we present an application of the previous results that shows how the projection of a game on a space of separable games may be useful in the analysis of approximate Nash equilibria of the original game. Indeed, the projection

$\Pi_{\mathcal{F}} u$ represents the closest game to u that is separable with respect to the FDH-graph $\mathcal{F}$. We can then adapt [31, Theorem 6.3] to relate the approximate equilibria of a game to the equilibria of the closest separable game with a given FDH-graph structure.

Proposition 5.2.1. *Let u be a game with player set $\mathcal{V}$, $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ be a FDH-graph and define*

$$\alpha := \|u - \Pi_{\mathcal{F}} u\|_{\mathcal{U}} = \sqrt{\langle u - \Pi_{\mathcal{F}} u, u - \Pi_{\mathcal{F}} u \rangle_{\mathcal{U}}}.$$

Then every ϵ_1 equilibrium of $\Pi_{\mathcal{F}} u$ is an ϵ_2 equilibrium of u for

$$\epsilon_2 \leq 2\alpha + \epsilon_1.$$

Despite its simplicity, Proposition 5.2.1 allows for some interesting applications. Suppose, for example, that the structure of interactions among players of a game of interest is very dense, meaning that every player is influenced by the actions of a large number of players. This is often the case when the game is the result of modeling a large, complex system. In this context, one can only derive a small computational advantage from the hypergraphical structure in the study of the relevant properties of the game. This happens because even the minimal FDH-graph of a game captures all dependencies among players, independently of their strength, giving a complete description of the interactions in the game, which may be too rich and detailed depending on the context.

Nevertheless, players of a game u are often known to interact over a network system, such as a communication infrastructure, which can be observed and modelled as an FDH-graph $\mathcal{F}$ and represents with good approximation the main dependencies among players. In such cases, the projection $\Pi_{\mathcal{F}} u$ captures the core structure of the game and, for each player, the main component of their utility function, while the orthogonal component $(I - \Pi_{\mathcal{F}})u$ collects the effect of minor interactions among players. The hypergraph structure of the projection $\Pi_{\mathcal{F}} u$ can be exploited to efficiently compute some properties of interest of the projected game [85, 77, 80], such as Nash equilibria, which can then be translated to the original game u thanks to Proposition 5.2.1.

We point out two main differences between the proposed decomposition (63) and other games' decompositions, like the potential-harmonic decomposition of [31] that has been discussed in Section 2.4.3. First, by Theorem 5.1.1 the projection of a game onto the space of separable games with a specified FDH-graph structure has an explicit characterization and can be computed efficiently. Second, the proposed framework allows to incorporate into the hypergraph $\mathcal{F}$ all the available knowledge about the system that is modelled by the game, thus obtaining an approximation that can be improved when new information is collected. The more

$\mathcal{F}$ is representative of the actual game structure, the more the projection $\Pi_{\mathcal{F}} u$ will be closer to the original game u, strengthening the result of Theorem 5.1.1.

Notice that the presence of extra dependencies in $\mathcal{F}$ with respect to $\mathcal{F}_u$ is not penalized by Proposition 5.2.1, as any hypergraph $\mathcal{F} \succeq \mathcal{F}_u$ will give the same accuracy on approximate Nash equilibria of u. However, there is a tradeoff on the size of the hypergraph $\mathcal{F}$, as superfluous hyperlinks will undermine the computational efficiency of computing Nash equilibria of $u_{\mathcal{F}}$.

These considerations are formalized by the following proposition for the case of pairwise separable games.

Proposition 5.2.2. *Let u be a pairwise separable game (34) and let $u_{\mathcal{G}}$ be its projection on a given graph $\mathcal{G}$.*

1. *the distance $\|u - u_{\mathcal{G}}\|$ only depends on the pairwise utilities u_{ij} over links (i,j) of $\mathcal{G}_u$ that are not part of $\mathcal{G}$.*
2. *if $u_{ij} = u_{hk}$ for all links $(i,j), (h,k) \in \mathcal{E}_u$ of $\mathcal{G}_u$, then*
$$\|u - u_{\mathcal{G}}\|^2 \propto |\{(i,j) \in \mathcal{E}_u \setminus \mathcal{E}\}|.$$

Proof. 1. Consider a pairwise separable game u on a graph $\mathcal{G}_u$ and let $u_{\mathcal{G}}$ be its projection on a given graph $\mathcal{G}$. To prove the first statement we show that

$$\|u - u_{\mathcal{G}}\|^2 = \sum_{(i,j) \in \mathcal{E}_u \setminus \mathcal{E}} dist(u_{ij}, \mathbb{R}^{\mathcal{X}}_{\{i\}})^2$$

where the distance $dist(u_{ij}, \mathbb{R}^{\mathcal{X}}_{\{i\}})$ of u_{ij} from the space of functions that only depend on the action of player i can be computed as

$$dist(u_{ij}, \mathbb{R}^{\mathcal{X}}_{\{i\}})^2 = |\mathcal{X}_{\mathcal{V} \setminus \{i,j\}}| \sum_{x_i \in \mathcal{A}_i} \sum_{x_j \mathcal{A}_j} \left(u_{ij}(x_i, x_j) - \frac{1}{|\mathcal{A}_j|} \sum_{y_j \in \mathcal{A}_j} u_{ij}(x_i, y_j) \right)^2 .$$

The utility of any player $i \in \mathcal{V}$ in $u_{\mathcal{G}}$ can be expressed as

$$\begin{aligned}
(u_{\mathcal{G}})_i(x) &= \frac{1}{\mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \sum_{y \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} u_i(x_{\mathcal{N}_i^\bullet(\mathcal{G})}, y) \\
&= \frac{1}{\mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \sum_{y \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \left(\sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \cap \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, x_j) + \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, y_j) \right) \\
&= \underbrace{\sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \cap \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, x_j)}_{(i)} + \underbrace{\frac{1}{\mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \sum_{y \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, y_j)}_{(ii)},
\end{aligned}$$

where the second term only depends on x_i. As a consequence, we can write

$$
\begin{aligned}
(u - u_{\mathcal{G}})_i(x) &= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u)} u_{ij}(x_i, x_j) - (i) - (ii) \\
&= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \cap \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, x_j) - (i) + \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, x_j) - (ii) \\
&= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, x_j) - \frac{1}{\mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \sum_{y \in \mathcal{X}_{\mathcal{V} \setminus \mathcal{N}_i^\bullet(\mathcal{G})}} \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} u_{ij}(x_i, y_j) \\
&= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} \left(u_{ij}(x_i, x_j) - \frac{1}{|\mathcal{A}_j|} \sum_{y_j \in \mathcal{A}_j} u_{ij}(x_i, y_j) \right) \\
&= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} \tilde{u}_{ij}(x_i, x_j)
\end{aligned}
$$

where $\tilde{u}_{ij}(x_i, x_j)$ is such that $\sum_{x_j \in \mathcal{A}_j} \tilde{u}_{ij}(x_i, x_j) = 0$. Formally, this means that $\tilde{u}_{ij} \in \mathbb{R}^{\mathcal{X}}_{\{i,j\}} \cap (\mathbb{R}^{\mathcal{X}}_{\{i\}})^\perp$. Moreover, for all $j \neq k \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})$, the spaces $\mathbb{R}^{\mathcal{X}}_{\{i,j\}} \cap (\mathbb{R}^{\mathcal{X}}_{\{i\}})^\perp$ and $\mathbb{R}^{\mathcal{X}}_{\{i,k\}} \cap (\mathbb{R}^{\mathcal{X}}_{\{i\}})^\perp$ are orthogonal, as

$$
\begin{aligned}
\langle \tilde{u}_{ij}, \tilde{u}_{ik} \rangle &= \sum_{x \in \mathcal{X}} \tilde{u}_{ij}(x_i, x_j) \tilde{u}_{ik}(x_i, x_k) \\
&= |\mathcal{X}_{\mathcal{V} \setminus \{i,j,k\}}| \sum_{x_i \in \mathcal{A}_i} \sum_{x_j \in \mathcal{A}_j} \tilde{u}_{ij}(x_i, x_j) \underbrace{\sum_{x_k \in \mathcal{A}_k} \tilde{u}_{ik}(x_i, x_k)}_{=0} = 0.
\end{aligned}
$$

Exploiting this orthogonality property, we can show that

$$
\begin{aligned}
\|u_i - (u_{\mathcal{G}})_i\|^2 &= \| \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} \tilde{u}_{ij} \|^2 \\
&= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} \|\tilde{u}_{ij}\|^2, \\
&= \sum_{j \in \mathcal{N}_i(\mathcal{G}_u) \setminus \mathcal{N}_i(\mathcal{G})} |\mathcal{X}_{\mathcal{V} \setminus \{i,j\}}| \sum_{x_i \in \mathcal{A}_i} \sum_{x_j \in \mathcal{A}_j} |\tilde{u}_{ij}(x_i, x_j)|^2.
\end{aligned}
$$

This shows that $\|u - u_{\mathcal{G}}\|^2$ only depends on the pairwise utilities u_{ij} over links (i, j) of $\mathcal{G}_u$ that are not part of $\mathcal{G}$ and that it decomposes linearly on links, so that it can be computed link by link. More precisely, each link $(i, j) \in \mathcal{E}_u \setminus \mathcal{E}$ contributes to $\|u - u_{\mathcal{G}}\|^2$ with a quantity

$$
\|\tilde{u}_{ij}\|^2 = |\mathcal{X}_{\mathcal{V} \setminus \{i,j\}}| \sum_{x_i \in \mathcal{A}_i} \sum_{x_j \mathcal{A}_j} \left(u_{ij}(x_i, x_j) - \frac{1}{|\mathcal{A}_j|} \sum_{y_j \in \mathcal{A}_j} u_{ij}(x_i, y_j) \right)^2.
$$

Notice that $\|\tilde{u}_{ij}\|$ is the distance of u_{ij} from the space of functions that only depend on the action of player i, i.e.,$\|\tilde{u}_{ij}\| = \|u_{ij} - \Pi_{\{i\}} u_{ij}\| = dist(u_{ij}, \mathbb{R}^{\mathcal{X}}_{\{i\}})$. This fact implies that we can express

$$\|u - u_{\mathcal{G}}\|^2 = dist(u, \mathcal{U}_{\mathcal{G}})^2 = \sum_{(i,j) \in \mathcal{E}_u \setminus \mathcal{E}} dist(u_{ij}, \mathbb{R}^{\mathcal{X}}_{\{i\}})^2.$$

2. In the case when $u_{ij} = u_{hk}$ for all links $(i,j),(h,k)$ of $\mathcal{G}_u$ the distance $dist(u_{ij}, \mathbb{R}^{\mathcal{X}}_{i})^2$ is independent on i, j. By denoting such distance with a constant $C \geq 0$, we see that $\|u - u_{\mathcal{G}}\|^2 = C|\{(i,j) \in \mathcal{E}_u \setminus \mathcal{E}\}|$, which shows that the distance between u and its graphical projection $u_{\mathcal{G}}$ decreases with the number of links of $\mathcal{G}_u$ that are not in $\mathcal{G}$.

□

5.3 APPENDIX: SUM OF PROJECTORS

We start with a general fact of linear algebra. Let V be a vector space equipped with a scalar product $<,>$. Given a subspace H of V, we denote by $\Pi_H : V \to V$ the orthogonal projector onto H. Two subspaces H, K of V are called *perpendicular* if $\Pi_H \Pi_K = \Pi_K \Pi_H$.

Lemma 5.3.1. *Let $\mathcal{F}$ be a family of subspaces of V that are pairwise perpendicular. Given $S \subseteq \mathcal{F}$, we denote by $\sum S$ and by $\cap S$, respectively the sum and the intersection of the subspaces in S. Then,*

$$\Pi_{\cap \mathcal{F}} = \prod_{H \in \mathcal{F}} \Pi_H, \quad \Pi_{\sum \mathcal{F}} = \sum_{\substack{S \subset \mathcal{F} \\ S \neq \emptyset}} (-1)^{|S|+1} \Pi_{\cap S} \tag{70}$$

Proof. We first prove that if H and K are two perpendicular subspaces, then $\Pi_H \Pi_K = \Pi_{H \cap K}$. Indeed notice that $T = \Pi_H \Pi_K$ is an orthogonal projector as

$$T^2 = \Pi_H \Pi_K \Pi_H \Pi_K = \Pi_H \Pi_H \Pi_K \Pi_K = T, \; T^* = \Pi_K^* \Pi_H^* = \Pi_K \Pi_H = \Pi_H \Pi_K = T$$

Since $T(V) \subseteq H \cap K$ and $T_{|H \cap K}$ coincides with the identity, we conclude that $T = \Pi_{H \cap K}$. A direct inductive argument now proves the first relation in (70). Concerning the second, first notice that

$$\mathcal{F}^{\perp} = \{H^{\perp} \mid H \in \mathcal{F}\}$$

is a family of pairwise perpendicular subspaces as the corresponding orthogonal projectors are $\Pi_{H^{\perp}} = I - \Pi_H$. Using standard orthogonality properties and the first relation in (70), we can now compute as follows

$$\Pi_{\sum \mathcal{F}} = I - \Pi_{\cap \mathcal{F}^{\perp}} = I - \prod_{H \in \mathcal{F}} \Pi_{H^{\perp}} = I - \prod_{H \in \mathcal{F}} (I - \Pi_H)$$

from which the second relation in (70) directly follows. □

Proof of Formula (66). We prove that the family of subspaces $\{\mathbb{R}^{\mathcal{X}}_{\mathcal{J}} \mid \mathcal{J} \in \mathcal{L}\}$ is a perpendicular family. Indeed, taken $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{V}$ the composed application $\Pi_{\mathcal{J}_1}\Pi_{\mathcal{J}_2}$ can be described as follows. We imagine configuration vectors split into four parts, corresponding to the four subsets of labels: $\mathcal{J}_1 \cap \mathcal{J}_2$, $\bar{\mathcal{J}}_1 = \mathcal{J}_1 \setminus \mathcal{J}_1 \cap \mathcal{J}_2$, $\bar{\mathcal{J}}_2 = \mathcal{J}_2 \setminus \mathcal{J}_1 \cap \mathcal{J}_2$, and $(\mathcal{J}_1 \cup \mathcal{J}_2)^c$ and we represent

$$\Pi_{\mathcal{J}_1}\Pi_{\mathcal{J}_2} f(x) = \frac{|\mathcal{X}_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}|}{|\mathcal{X}_{\mathcal{J}_1^c}||\mathcal{X}_{\mathcal{J}_2^c}|} \sum_{y_1 \in \mathcal{X}_{\bar{\mathcal{J}}_1}} \sum_{y_2 \in \mathcal{X}_{\bar{\mathcal{J}}_2}} \sum_{z \in \mathcal{X}_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}} f(x_{\mathcal{J}_1 \cap \mathcal{J}_2}, y_1, y_2, z)$$

The form of the right hand side implies that $\Pi_{\mathcal{J}_1}\Pi_{\mathcal{J}_2} = \Pi_{\mathcal{J}_2}\Pi_{\mathcal{J}_1}$.

Since, by definition, $\mathbb{R}^{\mathcal{X}}_{\mathcal{H}} = \sum_{\mathcal{J} \in \mathcal{L}} \mathbb{R}^{\mathcal{X}}_{\mathcal{J}}$, formula (66) is a direct consequence of Lemma 5.3.1. □

6

CORRELATED EQUILIBRIA OF SEPARABLE GAMES

In this chapter we investigate the structure of correlated equilibria in separable games, following the lines traced by [58] for graphical games, and we obtain both representational and computational results

As proved in [58], any correlated equilibrium P of a graphical game on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be represented as a local Markov network on $\mathcal{G}$ up to neighborhood equivalence, meaning that there exists a distribution Q, with the same marginals as P over the neighborhoods of $\mathcal{G}$, that is still a CE for the game and is a local Markov network on $\mathcal{G}$, i.e., it can be factorized over the neighborhoods $\mathcal{N}_i^\bullet$ of the game's graph $\mathcal{G}$ as $Q(x) = \frac{1}{Z}\left(\Pi_{i\in\mathcal{V}}\varphi_i(x_{\mathcal{N}_i^\bullet})\right)$ where the functions $\varphi_i : \mathcal{X}_{\mathcal{N}_i^\bullet} \to [0, \infty)$ are called local potentials and $Z = \sum_{x\in\mathcal{X}}\prod_{i\in\mathcal{V}}\varphi_i(x_{\mathcal{N}_i^\bullet})$ is a normalization factor. The authors employ this representational results to derive a polynomial time algorithm for computing correlated equilibria of graphical games, provided that the graph of the game is a tree. We underline that the polynomial complexity of the algorithm is measured with respect to the strict graphical representation of the input game, which is of size $O(n2^d)$ in case of binary games, where n is the number of players and d is the largest degree of the underlying graph. While this result improves on algorithms for normal form games, it is not satisfactory for games that allow for an even more compact representation, such as strict separable games. For example, strict pairwise separable games over a star graph can be represented with size $O(n)$, while the corresponding graphical representation has size $O(2^n)$, which is exponentially larger and comparable to the normal form size, which is $O(n2^n)$.

In this section we elaborate on these ideas and we show how these results can be extended to the setting of separable games. We then discuss the computational and descriptive advantages that result from the separable representation of games.

As shown in Proposition 2.4.3, correlated equilibria are preserved by strategic equivalence. For this reason, in this section we focus on normalized games. As discussed in Section 4.3, this assumption ensures that the separable representation of the games under analysis is compact and the complexity analysis of the presented algorithms is meaningful. The results we propose are applicable to a class of separable games associated to decomposable hypergraphs. We first present the theory in this general setting and we then focus on a relevant special case, that of pairwise separable games on trees.

6.1 REPRESENTATION OF CORRELATED EQUILIBRIA

The special structure of separable games allows to reformulate the CE condition (21) locally with respect to the game's underlying hypergraph.

Proposition 6.1.1. *Let u be a separable game over $\mathcal{F} = (\mathcal{V}, \mathcal{D})$. A joint distribution $P \in \Delta(\mathcal{X})$ is a correlated equilibrium for u if and only if the following linear constraints are satisfied: $\forall i \in \mathcal{V}, \forall a, b \in \mathcal{A}_i$*

$$\sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x_\mathcal{J}\in\mathcal{X}_\mathcal{J}} u_i^\mathcal{J}(a, x_\mathcal{J}) P(a, x_\mathcal{J}) \geq \sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x_\mathcal{J}\in\mathcal{X}_\mathcal{J}} u_i^\mathcal{J}(b, x_\mathcal{J}) P(a, x_\mathcal{J}). \tag{71}$$

Proof. The fact that $P \in CE(u)$ is equivalent to each of the following inequalities, valid for all $i \in \mathcal{V}$ and $a, b \in \mathcal{A}_i$:

$$\begin{aligned}
&\sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x\in\mathcal{X}} u_i^\mathcal{J}(a, x_\mathcal{J}) P(x|x_i = a) \\
&\qquad \geq \sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x\in\mathcal{X}} u_i^\mathcal{J}(b, x_\mathcal{J}) P(x|x_i = a) \\
\Leftrightarrow &\sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x_i, x_\mathcal{J}\in\mathcal{X}_i\times\mathcal{X}_\mathcal{J}} u_i^\mathcal{J}(a, x_\mathcal{J}) P(x_i, x_\mathcal{J}|x_i = a) \\
&\qquad \geq \sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x_i, x_\mathcal{J}\in\mathcal{X}_i\times\mathcal{X}_\mathcal{J}} u_i^\mathcal{J}(b, x_\mathcal{J}) P(x_i, x_\mathcal{J}|x_i = a) \\
\Leftrightarrow &\sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x_\mathcal{J}\in\mathcal{X}_\mathcal{J}} u_i^\mathcal{J}(a, x_\mathcal{J}) P(a, x_\mathcal{J}) \\
&\qquad \geq \sum_{(i,\mathcal{J})\in\mathcal{D}} \sum_{x_\mathcal{J}\in\mathcal{X}_\mathcal{J}} u_i^\mathcal{J}(b, x_\mathcal{J}) P(a, x_\mathcal{J}).
\end{aligned}$$

□

The previous result shows that the fact that a joint distribution $P \in \Delta(\mathcal{X})$ satisfies the CE condition for a $\mathcal{F}$-separable game depends only on its marginals over the hyperlinks of $\mathcal{H}^\mathcal{F}$. It is then natural to define the following equivalence notion over $\Delta(\mathcal{X})$, that does not distinguish among distributions whose hyperlink marginals coincide.

Definition 6.1.2 (Hyperlink equivalence)**.** *For a H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, two distributions P and Q in $\Delta(\mathcal{X})$ are hyperlink equivalent, denoted $P \equiv_\mathcal{H} Q$, if*

$$\forall \mathcal{J} \in \mathcal{L}, \forall x \in \mathcal{X}, \quad P(x_\mathcal{J}) = Q(x_\mathcal{J}) \tag{72}$$

As a consequence of Proposition 6.1.1, hyperlink equivalence $\equiv_{\mathcal{H}^{\mathcal{F}}}$ preserves correlated equilibria of $\mathcal{F}$-separable games. In each class of hyperlink equivalent CE, a special role is played by those which can be factorized over hyperlinks. Formally,

Definition 6.1.3. *A distribution $P \in \Delta(\mathcal{X})$ factorizes over an hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ if it can be expressed as*

$$P(x) = \prod_{\mathcal{J} \in \mathcal{L}} \varphi_{\mathcal{J}}(x_{\mathcal{J}}).$$

Factorizable CE are particularly interesting for three main reasons.

First, by a maximum entropy argument it can be shown that, for any $\mathcal{F}$-separable game u, every hyperlink equivalence class of $CE(u)$ has a member, factorizable over $\mathcal{H}^{\mathcal{F}}$, that is its maximum entropy element, i.e., the one that makes the fewest assumptions about the system [21, 56, 78]. This is a consequence of the following Lemma, whose proof, inspired by [78], can be found at the end of this section.

Lemma 6.1.4. *For all H-graphs $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ and for all distributions P over $\mathcal{X}$, there exists a distribution $Q \equiv_{\mathcal{H}} P$ that can be factorized over $\mathcal{H}$.*

In other words, as a consequence of Lemma 6.1.4, up to hyperlink equivalence any correlated equilibria of an $\mathcal{F}$-separable game can be factorized over $\mathcal{H}^{\mathcal{F}}$. By the same token, factorizable distributions satisfy the highest number of conditional independency statements between players' actions, i.e., they don't assume any further correlation among players than those following directly from the hyperlink marginal distributions common to all class members.

Moreover, given a correlated equilibrium $P \in CE(u)$, constructing a factorizable hyperlink equivalent equilbrium $Q \equiv_{\mathcal{H}^{\mathcal{F}}} P$ can be done efficiently whenever $\mathcal{H}^{\mathcal{F}}$ is decomposable. Under this assumption, the factorizable member of the class is unique, showing that it is a natural representative for the entire class.

More generally, one can build a factorizable correlated equilibrium starting from any family of distributions over hyperlinks, provided they are consistent in the sense of Definition 2.3.9.

All these facts are consequences of the following result.

Theorem 6.1.5. *Consider an $\mathcal{F}$-separable game u, assume that the hypergraph $\mathcal{H}^{\mathcal{F}} = (\mathcal{V}, \mathcal{L})$ is decomposable and let $\{P_{\mathcal{J}} \in \Delta(\mathcal{X}_{\mathcal{J}})\}_{\mathcal{J} \in \mathcal{L}}$ be any family of consistent local distributions, i.e., such that $\forall \mathcal{J}, \mathcal{K} \in \mathcal{L}$, $P_{\mathcal{J}}(x_{\mathcal{J} \cap \mathcal{K}}) = P_{\mathcal{K}}(x_{\mathcal{J} \cap \mathcal{K}})$, which satisfy the local CE condition (71). Then, there exists a unique joint distribution Q, defined by*

$$Q(x) = \frac{\prod_{\mathcal{J} \in \mathcal{L}} P_{\mathcal{J}}(x_{\mathcal{J}})}{\prod_{\mathcal{T} \in \mathcal{S}} P_{\mathcal{T}}(x_{\mathcal{T}})^{\nu(\mathcal{T})}} \tag{73}$$

where $\mathcal{S}$ are the separators of $\mathcal{H}^{\mathcal{F}}$, that factorizes over $\mathcal{H}^{\mathcal{F}}$ and has the family as its hyperlink marginals, i.e., $Q(x_{\mathcal{J}}) = P_{\mathcal{J}}(x_{\mathcal{J}})$ for all $\mathcal{J} \in \mathcal{L}$.

Proof. This is a direct consequence of Lemma 2.3.10 □

Finally, given any objective function over the CE of an $\mathcal{F}$-separable game that only depends on their marginal distributions over hyperlinks of $\mathcal{H}^{\mathcal{F}}$, a maximizer can always be taken to be factorizable over $\mathcal{H}^{\mathcal{F}}$. The assumption for the objective function to depend only on hyperlink marginals is indeed quite natural when dealing with normalized separable games, as for example the expected payoff of players has this property. Indeed, as shown in Theorem 4.2.7, $\mathcal{F}$-separability of a normalized game u implies that for any player $i \in \mathcal{V}$, u_i admits the following decomposition

$$u_i(x) = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_{i,\mathcal{J}}(x_i, x_{\mathcal{J}}), \quad \forall i \in \mathcal{V}, \forall x \in \mathcal{X}, \tag{74}$$

where no non-strategic terms appear. It then follows that the expected payoff with respect to any distribution $P \in \Delta(\mathcal{X})$ can be expressed as

$$\begin{aligned}
\mathbb{E}_{x\sim P}\left[u_i(x)\right] &= \mathbb{E}_{x\sim P}\left[\sum_{(i,\mathcal{J}) \in \mathcal{D}} u_{i,\mathcal{J}}(x_i, x_{\mathcal{J}})\right] \\
&= \sum_{(i,\mathcal{J}) \in \mathcal{D}} \mathbb{E}_{x\sim P}\left[u_{i,\mathcal{J}}(x_i, x_{\mathcal{J}})\right] \\
&= \sum_{(i,\mathcal{J}) \in \mathcal{D}} \sum_{x \in \mathcal{X}} u_{i,\mathcal{J}}(x_i, x_{\mathcal{J}}) P(x) \\
&= \sum_{(i,\mathcal{J}) \in \mathcal{D}} \sum_{x_i, x_{\mathcal{J}} \in \mathcal{X}_i \times \mathcal{X}_{\mathcal{J}}} u_{i,\mathcal{J}}(x_i, x_{\mathcal{J}}) P(x_{\{i\}\cup\mathcal{J}}),
\end{aligned}$$

which shows it is linear in the hyperlink marginals, since $\{i\} \cup \mathcal{J}$ is an hyperlink of $\mathcal{H}^{\mathcal{F}}$. In the case of linear objective function we present an efficient algorithm to retrieve such maximiser under the assumption that $\mathcal{H}^{\mathcal{F}}$ is decomposable.

To obtain such efficient algorithm, we adapt the construction of [58]: for a given normalized separable game u over $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, we denote $\mathcal{H}^{\mathcal{F}} = (\mathcal{V}, \mathcal{L})$ and we define a set of linear constraints over the variables $P_{\mathcal{J}}(x_{\mathcal{J}}) \geq 0$ for $\mathcal{J} \in \mathcal{L}$ and $x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}$

$$\begin{aligned}
&\forall i \in \mathcal{V}, \forall a, b \in \mathcal{A}_i, \\
&\sum_{(i,\mathcal{J}) \in \mathcal{D}} \sum_{x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}} u_i^{\mathcal{J}}(a, x_{\mathcal{J}}) P_{\{i\} \cup \mathcal{J}}(a, x_{\mathcal{J}}) \geq \sum_{(i,\mathcal{J}) \in \mathcal{D}} \sum_{x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}} u_i^{\mathcal{J}}(b, x_{\mathcal{J}}) P_{\{i\} \cup \mathcal{J}}(a, x_{\mathcal{J}}) \\
&\forall \mathcal{J}, \mathcal{K} \in \mathcal{L}, P_{\mathcal{J}}(x_{\mathcal{J} \cap \mathcal{K}}) = P_{\mathcal{K}}(x_{\mathcal{J} \cap \mathcal{K}}), \\
&\forall \mathcal{J} \in \mathcal{L}, \sum_{x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}} P_{\mathcal{J}}(x_{\mathcal{J}}) = 1.
\end{aligned} \tag{75}$$

By Proposition 6.1.1, this set of constraints is necessary for the existence of a correlated equilibrium $P \in CE(u)$ such that $\forall \mathcal{J} \in \mathcal{L}$ and $\forall x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}$, $P(x_{\mathcal{J}}) = P_{\mathcal{J}}(x_{\mathcal{J}})$. When $\mathcal{H}^{\mathcal{F}}$ is a decomposable, the constraints are also sufficient, resulting in the following.

Proposition 6.1.6. *Let u be a $\mathcal{F}$-separable game such that $\mathcal{H}^{\mathcal{F}}$ is decomposable. Consider the problem of maximizing over $CE(u)$ a linear objective function that only depends of the CE marginals over hyperlinks of $\mathcal{H}^{\mathcal{F}}$. Then an optimal solution Q can be obtained in polynomial time by first solving a linear programming with the provided objective function and the constraints (75) and by then applying equation (73) to the result. In particular, the resulting correlated equilibrium factorizes over $\mathcal{H}^{\mathcal{F}}$.*

6.2 DISCUSSION AND EXAMPLES

In this section we comment on our results and we show how the separable representation of normalized games allows to refine the analogous results from [58] based on the graphical representation. As previously mentioned, a very relevant class of games for which our results apply is that of pairwise separable games on tree graphs, as for such games the minimal FDH-graph $\mathcal{F}$ is such that $\mathcal{H}^{\mathcal{F}}$ is decomposable. In discussing the results we then focus on this class of games, for which the benefit of our theory can be easily appreciated.

Consider any normalized pairwise separable game that can be either represented as a $\mathcal{G}$-graphical game on a tree or as a $\mathcal{F}$-separable game where $\mathcal{G}^{\mathcal{F}}$ is a tree (see Figure 13 for an example).

The separable representation of games yields a finer factorization of CE than the graphical. Indeed, Proposition 6.1.5 and the related discussion implies that, up to hyperlink equivalence $\equiv_{\mathcal{H}^{\mathcal{F}}}$, all correlated equilibria of u can be factorized over links of $\mathcal{G}$. As depicted in Figure 13 this is in general a much finer factorization than the one on neighborhoods of $\mathcal{G}$, which is presented in [58]. This has major consequences both on the computational and descriptive side. Indeed, the finer

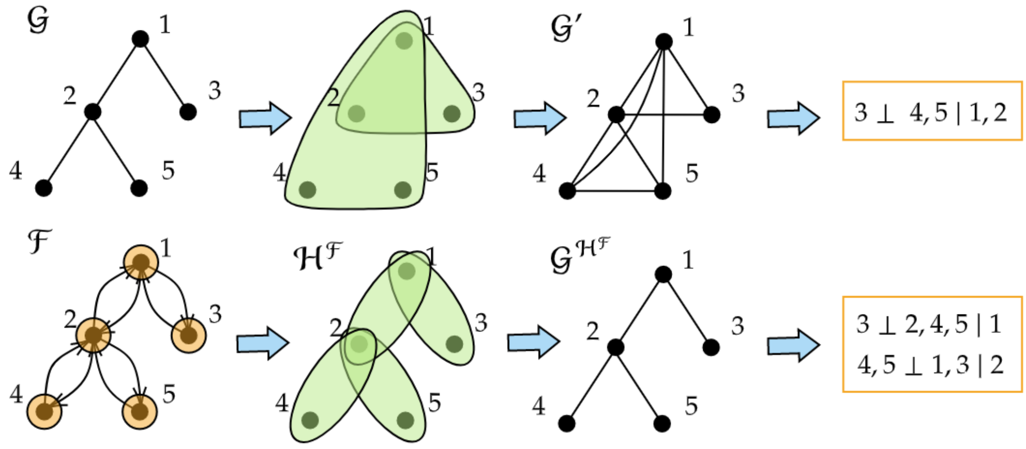


Figure 13: Conditional independency relationships arising among players in factorizable CE of a pairwise game within the graphical and separable representation.

factorization results in a smaller number of variables in the linear program (75), which makes the corresponding algorithm efficient for separable games. Moreover, a finer factorization of CE corresponds to a larger number of conditional independency relationships among players at equilibrium. More precisely, by adopting the separable representation, one can read the conditional independencies among players at a factorizable equilibrium on the graph $\mathcal{G}^{\mathcal{H}^{\mathcal{F}}}$, which is such that each player i is independent on $\mathcal{V} \setminus \mathcal{N}_i^{\bullet}$ conditioned on $\mathcal{N}_i$ (see Figure 13). Similarly, with the graphical representation, conditional independencies are encoded in the graph $\mathcal{G}' := \mathcal{G}^{\mathcal{F}^{\leftrightarrow}}$. However, in general $\mathcal{G}^{\mathcal{H}^{\mathcal{F}}}$ contains a much smaller number of links than $\mathcal{G}'$. By ignoring the pairwise nature of the game, one can only compute efficiently (with respect to the more verbose graphical representation) correlated equilibria that introduce unnecessary correlations among players. Conversely, the separable representation allows to obtain efficiently (with respect to the more compact input representation) correlated equilibria that only include the essential dependencies among players.

This becomes apparent in cases when adopting the graphical representation yields a trivial factorization of CE, corresponding to assuming arbitrary correlations among all players and resulting in having no improvements on both the computational and descriptive side with respect to the standard normal form representation. An example of this situation is illustrated in Figure 14 for a pairwise separable game on a star graph $\mathcal{G}$, which highlights the advantages of the separable representation.

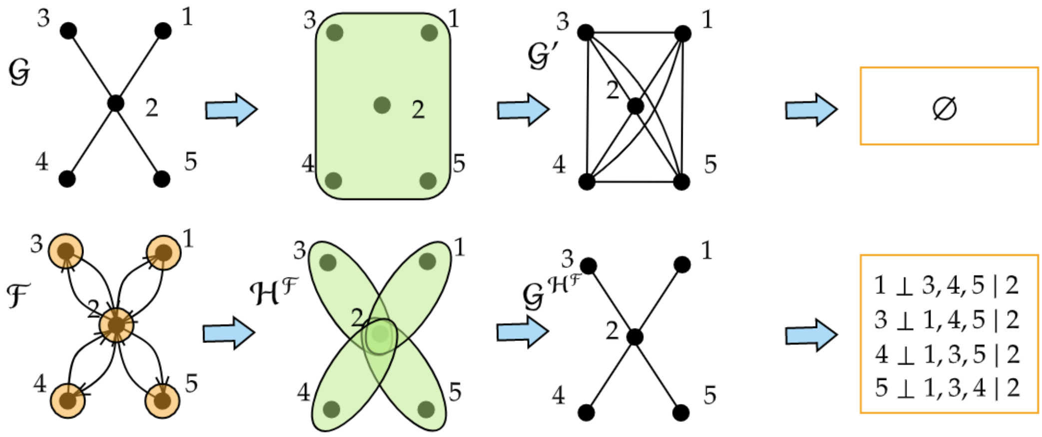


Figure 14: Pairwise game for which the graphical representation yields a trivial CE factorization result, compared to the corresponding separable representation.

6.3 APPENDIX: PROOF OF LEMMA 6.1.4

Proof. To prove the statement we construct a distribution Q that is consistent with the players hyperlink marginals under P and can be also factorized over $\mathcal{H} = (\mathcal{V}, \mathcal{L})$. More precisely, we show that the maximum entropy distribution satisfying the marginals constraints can be factorized over $\mathcal{H}$. To do this, we define the entropy function $H(Q) = \sum_{x \in \mathcal{X}} Q(x) \log(\frac{1}{Q(x)})$ and the following constrained maximum entropy problem:

$$Q^* = \arg\max_{Q} H(Q) = \arg\max_{Q} \sum_{x \in \mathcal{X}} Q(x) \log(\frac{1}{Q(x)}) \tag{76}$$

$$\text{s.t.} \quad Q(x_{\mathcal{J}}) = P(x_{\mathcal{J}}) \quad \forall \mathcal{J} \in \mathcal{L}, \forall x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}$$

$$\sum_{x \in \mathcal{X}} Q(x) = 1$$

Notice that the objective function $H(Q)$ is strictly concave, all constraints are linear and that the feasible set is non-empty, as the distribution P itself is a feasible solution. This guarantees that the above optimization problem admits a unique solution Q^*. Such Q^* is the maximum entropy distribution over distributions that are hyperlink equivalent to P with respect to $\mathcal{H}$.

Moreover, observe that for all probability distributions Q, the non-negativity property implies that $\forall \mathcal{J} \subset \mathcal{V}$ and for all $x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}$

$$Q(x_{\mathcal{J}}) = \sum_{\substack{y \in \mathcal{X} \\ y_{\mathcal{J}} = x_{\mathcal{J}}}} Q(y) = 0 \quad \Leftrightarrow \quad \forall y \in \mathcal{X} \text{ s.t. } y_{\mathcal{J}} = x_{\mathcal{J}},\ Q(y) = 0.$$

This, together with the marginals constraints, implies that $\forall \mathcal{J} \in \mathcal{L}$

$$P(x_{\mathcal{J}}) = 0 \quad \Leftrightarrow \quad Q^*(x_{\mathcal{J}}) = 0 \quad \Rightarrow \quad Q^*(x) = 0.$$

As a consequence, we can remove from the optimization problem all variables $Q(x)$ such that $P(x_{\mathcal{J}}) = 0$ for some $\mathcal{J} \in \mathcal{L}$, by setting them to zero.

Formally, we define the two sets

$$\mathcal{X}^+ = \{x \in \mathcal{X} : \forall \mathcal{J} \in \mathcal{L}, P(x_{\mathcal{J}}) > 0\}$$
$$\mathcal{X}^0 = \{x \in \mathcal{X} : \exists \mathcal{J} \in \mathcal{L}, P(x_{\mathcal{J}}) = 0\}$$

and we set $Q(x) = 0$ for all $x \in \mathcal{X}^0$. Then, to find the optimal Q^* we restate the problem within the Lagrange multiplier formalism. By introducing the Lagrange multipliers $\lambda = (\lambda_{\mathcal{J},x_{\mathcal{J}}})_{\mathcal{J}\in\mathcal{L},x_{\mathcal{J}}\in\mathcal{X}_{\mathcal{J}}}$ and β we can reformulate the optimization problem as

$$\begin{aligned} Q^* &= \arg\max_{\substack{Q:\\ \forall x\in\mathcal{X}^0,\, Q(x)=0}} \max_{\lambda,\beta} \quad L(Q,\lambda,\beta) \\ &\equiv \arg\max_{\substack{Q:\\ \forall x\in\mathcal{X}^0,\, Q(x)=0}} \max_{\lambda,\beta} \quad H(Q) \\ &\quad + \sum_{\mathcal{J}\in\mathcal{L}} \sum_{x_{\mathcal{J}}\in\mathcal{X}_{\mathcal{J}}} \lambda_{\mathcal{J},x_{\mathcal{J}}} \left(Q(x_{\mathcal{J}}) - P(x_{\mathcal{J}})\right) \\ &\quad + \beta \sum_{x\in\mathcal{X}} \left(Q(x) - 1\right). \end{aligned} \tag{77}$$

The optimal solution of (76) can be obtained by finding the stationary point $(Q^*, \lambda^*, \beta^*)$ of the Lagrangian function $L(Q, \lambda, \beta)$. We start by computing

$$\begin{aligned} \frac{\partial L}{\partial Q(x)}(Q,\lambda,\beta) &= \frac{\partial H}{\partial Q(x)}(Q) + \sum_{\mathcal{J}\in\mathcal{L}} \sum_{y_{\mathcal{J}}\in\mathcal{X}_{\mathcal{J}}} \lambda_{\mathcal{J},y_{\mathcal{J}}} \frac{\partial Q(y_{\mathcal{J}})}{\partial Q(x)} + \beta \\ &= -1 - \log Q(x) + \sum_{\mathcal{J}\in\mathcal{L}} \lambda_{\mathcal{J},x_{\mathcal{J}}} + \beta \end{aligned} \tag{78}$$

where the last equality holds for $Q(x) > 0$ and follows from the fact that

$$\frac{\partial Q(y_{\mathcal{J}})}{\partial Q(x)} = \begin{cases} 1 & \text{if } y_{\mathcal{J}} = x_{\mathcal{J}} \\ 0 & \text{otherwise.} \end{cases}$$

Then, either $Q^*(x) = 0$ or $Q^*(x) > 0$ and by imposing $\frac{\partial L}{\partial Q(x)}(Q^*, \lambda^*, \beta^*) = 0$ in (78) we have that

$$\begin{aligned} &\log Q^*(x) = -1 + \beta^* + \sum_{\mathcal{J}\in\mathcal{L}} \lambda^*_{\mathcal{J},x_{\mathcal{J}}} \\ \Longrightarrow\ & Q^*(x) = e^{-1+\beta^*} e^{\sum_{\mathcal{J}\in\mathcal{L}} \lambda^*_{\mathcal{J},x_{\mathcal{J}}}} = e^{-1+\beta^*} \prod_{\mathcal{J}\in\mathcal{L}} e^{\lambda^*_{\mathcal{J},x_{\mathcal{J}}}}. \end{aligned}$$

This shows that either $Q^*(x) = 0$ or $Q^*(x)$ can be factorized over $\mathcal{H}$. To extend the factorization of Q^* to all $\mathcal{X}$ we show that $Q^*(x) = 0$ only for $x \in \mathcal{X}^0$ by proving that $Q^*(x) > 0$ for all $x \in \mathcal{X}^+$. Consider $x \in \mathcal{X}^+$ and a measure Q such that $Q(x) = 0$. We show that for any λ and β, $L(Q, \lambda, \beta)$ is not optimal. Indeed, for all $\epsilon > 0$ we define the measure

$$Q^\epsilon(y) = \begin{cases} \epsilon & \text{if } y = x \\ Q(y) & \text{otherwise} \end{cases}$$

and we have that

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{\mathrm{d}L}{\mathrm{d}\epsilon}(Q^\epsilon, \lambda, \beta) &= \lim_{\epsilon \to 0} \sum_{y \in \mathcal{X}} \frac{\partial L}{\partial Q(y)}(Q^\epsilon, \lambda, \beta) \frac{\mathrm{d}Q^\epsilon(y)}{\mathrm{d}\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\partial L}{\partial Q(x)}(Q^\epsilon, \lambda, \beta) \frac{\mathrm{d}Q^\epsilon(x)}{\mathrm{d}\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\partial L}{\partial Q(x)}(Q^\epsilon, \lambda, \beta) \\
&= \lim_{\epsilon \to 0} -1 - \log Q^\epsilon(x) + \sum_{\mathcal{J} \in \mathcal{L}} \lambda_{\mathcal{J}, x_\mathcal{J}} + \beta \\
&= \lim_{\epsilon \to 0} -1 - \log \epsilon + \sum_{\mathcal{J} \in \mathcal{L}} \lambda_{\mathcal{J}, x_\mathcal{J}} + \beta = +\infty.
\end{aligned}$$

Then, for a sufficiently small $\tilde{\epsilon} > 0$ we have that

$$\frac{\mathrm{d}L}{\mathrm{d}\epsilon}(Q^\epsilon, \lambda, \beta)|_{\epsilon = \tilde{\epsilon}} > 0.$$

Moreover, $L(Q^\epsilon, \lambda, \beta)$ is a concave function of ϵ, since L — being the sum of H and affine terms in Q — is a concave function of Q, and Q^ϵ is an affine function of ϵ, namely $Q^\epsilon = Q + \epsilon \delta_x$. It follows that $L(Q^{\tilde{\epsilon}}, \lambda, \beta) > L(Q, \lambda, \beta)$, so that Q is not a maximizer of $L(\cdot, \lambda, \beta)$. We have concluded that

$$Q^*(x) = \begin{cases} 0 & \text{if } x \in \mathcal{X}^0 \\ e^{-1+\beta^*} \prod_{\mathcal{J} \in \mathcal{L}} e^{\lambda^*_{\mathcal{J}, x_\mathcal{J}}} & \text{if } x \in \mathcal{X}^+, \end{cases}$$

which, by the definition of $\mathcal{X}^0$, can be rewritten as

$$Q^*(x) = e^{-1+\beta^*} \prod_{\mathcal{J} \in \mathcal{L}} \mathbb{I}_{P(x_\mathcal{J}) \neq 0} e^{\lambda^*_{\mathcal{J}, x_\mathcal{J}}}$$

where the indicator function $\mathbb{I}_{P(x_\mathcal{J}) \neq 0}$ is 1 if its condition $P(x_\mathcal{J}) \neq 0$ is verified, while it is 0 otherwise. This concludes the proof, as it shows that Q^* is both factorizable over $\mathcal{H}$ and it is hyperlink equivalent to P with respect to $\mathcal{H}$. □

7

SEPARABLE POTENTIAL GAMES

Separability of games reflects on their properties, which are often shaped by the hypergraph structure of the game itself. We already discussed this phenomenon in Chapter 6 where we drew a connection between separability of games and the structure of their correlated equilibria. This effect is even stronger when we focus on the class of separable potential games, which has a leading role in game theory. We devote this chapter to present in detail the structural properties of separable potential games and to discuss some of their many implications. We show that potential games are characterized by a symmetry of strategic interaction that is effectively captured in the formalism of separable games, as it simply translates in their minimal FDH-graph being undirected. We elaborate on the meaning of this result in Section 7.1 and we give a probabilistic interpretation of it in Section 7.2. We then collocate the structural result on potential games in the general theory of separable games, obtaining a characterization of the potential part of any separable game in Section 7.3. Finally in Section 7.4 we suggest how our result can be usefully exploited by briefly describing an application to the analysis of better response paths in separable potential games.

7.1 THE STRUCTURE OF POTENTIAL GAMES

In this section we focus on potential games and study how their separability is intertwined with the separability of the corresponding potential functions.

In general, the minimal graph associated to a game can be either directed or undirected. Corollary 7.1.3 shows that this is not the case when we restrict to potential games, which only exhibit undirected minimal graphs and thus are characterized by symmetry of interactions between players. Theorem 7.1.1 is an extension of this result, showing that the symmetry of interactions in potential games concerns not only binary but also higher order interactions.

We then derive, as a corollary, results on graphical potential games first appeared in [14] and we provide an alternative proof of the Hammersley-Clifford theorem for Markov random fields.

Theorem 7.1.1. *Let u in $\mathcal{U}$ be a potential game with potential function ϕ. Then, the minimal FDH-graph of u is the undirected FDH-graph associated to the minimal H-graph of ϕ, i.e.,*

$$\mathcal{F}_u = \mathcal{F}^{\mathcal{H}_\phi} \tag{79}$$

Proof. As discussed in Section 2.4, every potential game u in $\mathcal{U}$ is strategically equivalent to a game u^ϕ where all players' utilities equal the potential function ϕ. Since ϕ is $\mathcal{H}_\phi$-separable, with $\mathcal{H}_\phi = (\mathcal{V}, \mathcal{L}_\phi)$, we can write

$$u_i(x) = \phi(x) + n_i(x_{-i}) = \sum_{\mathcal{K} \in \mathcal{L}_\phi} \phi_{\mathcal{K}}(x_{\mathcal{K}}) + n_i(x_{-i}) \,, \tag{80}$$

for some function $n_i : \mathcal{X}_{-i} \to \mathbb{R}$. This shows that u is $\mathcal{F}^{\mathcal{H}_\phi}$-separable, therefore $\mathcal{F}_u \preceq \mathcal{F}^{\mathcal{H}_\phi}$.

Consider now the local H-graphs $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$, for i in $\mathcal{V}$, associated to the minimal FDH-graph $\mathcal{F}_u$ in the sense of (40). As discussed in Section 3.1, every utility function u_i is $\mathcal{H}_i$-separable and thus from the first equality in (80) we get that also ϕ is $\mathcal{H}_i$-separable, for every i in $\mathcal{V}$. By definition, every hyperlink $(i, \mathcal{J})$ of the FDH-graph $\mathcal{F}^{\mathcal{H}_\phi}$ is such that $\{i\} \cup \mathcal{J}$ in $\mathcal{L}_\phi$ is an undirected hyperlink of $\mathcal{H}_\phi$. Since the potential function ϕ is $\mathcal{H}_i$-separable and $\mathcal{H}_\phi$ is the minimal H-graph of ϕ, this implies that there exists $\mathcal{K}$ in $\mathcal{L}_i$ such that $\{i\} \cup \mathcal{J} \subseteq \mathcal{K}$. By the way the local H-graph $\mathcal{H}_i$ is defined, it follows that necessarily $(i, \mathcal{K} \setminus \{i\})$ in $\mathcal{D}_u$ is a hyperlink of the FDH-graph $\mathcal{F}_u$. We have thus just proved that $\mathcal{F}^{\mathcal{H}_\phi} \preceq \mathcal{F}_u$. The claim then follows as we had already shown that $\mathcal{F}_u \preceq \mathcal{F}^{\mathcal{H}_\phi}$. □

Theorem 7.1.1 is a structural result that characterizes the minimal separability property of potential games in terms of the minimal H-graph of their potential functions. It tells that the minimal FDH-graph $\mathcal{F}$ of potential games is undirected, meaning that if the utility of a player i depends jointly on the actions of players j and k in $\mathcal{F}$, then j and k are jointly dependent on each other and on i (see Figure 15).

Example 7.1.2 (Hyper-coordination game)**.** *Hyper-coordination games are a family of synchronisation games on hypergraphs [81] that model high-order coordination behavior of players. They are an extension of network coordination games on graphs presented in Example 2.4.8 to the setting of hypergraphs: players, corresponding to nodes of an H-graph, aim at simultaneously coordinating on some action with multiple* groups *of players, represented by hyperlinks. When all members of an hyperlink choose the same action, each player receives a positive payoff, which is additively combined with the ones deriving from all hyperlinks the player participates in. Formally, given an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, an action space $\mathcal{A}$ and a weight function w that associates to any action $a \in \mathcal{A}$ and hyperlink*

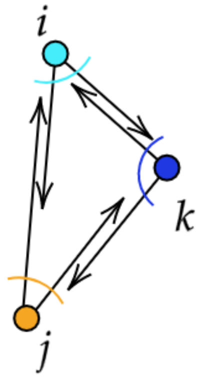


Figure 15: Representation of the symmetry of strategic interactions in potential games, as described by the undirected minimal FDH-graph of a the game $\mathcal{F}$. For example, if some player i depends jointly on the actions of players j and k in $\mathcal{F}$, than j and k are jointly dependent on each other and on i.

$\mathcal{K} \in \mathcal{L}$ a positive integer weight $w(a, \mathcal{K}) > 0$, the coordination game on $\mathcal{H}$ with player set $\mathcal{V}$ is defined by the utility functions:

$$u_i(x) = \sum_{\substack{\mathcal{K} \in \mathcal{L} \\ i \in \mathcal{K}}} \bar{w}(x, \mathcal{K}), \qquad \forall i \in \mathcal{V}, \forall x \in \mathcal{X} = \mathcal{A}^{\mathcal{V}} \tag{81}$$

where

$$\bar{w}(x, \mathcal{K}) = \begin{cases} 0 & \text{if } \exists j, k \in \mathcal{K} : x_j \neq x_k \\ w(x_j, \mathcal{K}) & \text{for any } j \in \mathcal{K}, \text{ otherwise.} \end{cases}$$

We will say that a coordination game is homogeneous if the weight function w is not dependent on the hyperlink $\mathcal{K}$ but only on the action a, and in such case we will again denote it by $w(a)$. As shown in [81, Lemma 6], every hyper-coordination game on an H-graph is potential, with potential function

$$\phi(x) = \sum_{\mathcal{K} \in \mathcal{L}} \bar{w}(x, \mathcal{K}), \tag{82}$$

with minimal H-graph $\mathcal{H}_\phi = \overline{\mathcal{H}}$.

It can be directly verified from the definition that, according to Theorem 7.1.1, a hyper-coordination game on a H-graph $\mathcal{H}$ is minimally separable with respect to the FDH-graph $\mathcal{F}_u = \mathcal{F}^{\overline{\mathcal{H}}}$.

Both the symmetry property and the relation between a game and its potential function can be translated in the graphical setting. Indeed, the previous result implies the following relation between the minimal graph of a potential game and the separability of the potential function. Given an undirected graph $\mathcal{G}$ we denote by $\mathcal{C}l(\mathcal{G})$ the set of maximal cliques in $\mathcal{G}$, and let $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l} = (\mathcal{V}, \mathcal{C}l(\mathcal{G}))$ be the cliques H-graph of $\mathcal{G}$.

Corollary 7.1.3. *Let u in $\mathcal{U}$ be a potential game with potential function ϕ. Then, the minimal graph $\mathcal{G}_u$ associated with u is undirected. Moreover, u is a $\mathcal{G}$-game for an undirected graph $\mathcal{G}$ if and only if its potential function ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable.*

Proof. Consider the minimal FDH-graph $\mathcal{F}_u = (\mathcal{V}, \mathcal{D}_u)$ of u and the minimal H-graph $\mathcal{H}_\phi = (\mathcal{V}, \mathcal{L}_\phi)$ of ϕ. It follows from Corollary 4.1.8 and relation (79) that $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u) = \mathcal{G}^{\mathcal{F}_u}$ is the graph associated to the undirected FDH-graph $\mathcal{F}_u = \mathcal{F}^{\mathcal{H}_\phi}$ and thus it is itself undirected.

Suppose now that u is a $\mathcal{G}$-game for some undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $\mathcal{K}$ in $\mathcal{L}_\phi$. Then, for every i in $\mathcal{K}$, (79) implies that $(i, \mathcal{K} \setminus \{i\})$ belongs to $\mathcal{D}_u$ and thus (i, j) belongs to $\mathcal{E}_u$ for every $i \neq j$ in $\mathcal{K}$. This says that $\mathcal{K}$ is a clique in $\mathcal{G}_u$ and thus also in $\mathcal{G}$, since $\mathcal{G}_u \preceq \mathcal{G}$. Therefore, $\mathcal{H}_\phi \preceq \mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$, thus showing that ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable.

Conversely, if ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable, then necessarily $\mathcal{H}_\phi \preceq \mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$, so that that every undirected hyperlink in $\mathcal{L}_\phi$ is contained in a clique of $\mathcal{G}$. By Corollary 4.1.8, the minimal graph $\mathcal{G}_u$ is the graph associated with the FDH-graph $\mathcal{F}_u$ and, by Theorem 7.1.1, $\mathcal{F}_u = \mathcal{F}^{\mathcal{H}_\phi}$ is undirected. It then follows that $\mathcal{G}_u = \mathcal{G}^{\mathcal{F}^{\mathcal{H}_\phi}} \preceq \mathcal{G}^{\mathcal{F}^{\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}}} = \mathcal{G}$, thus showing that u is a $\mathcal{G}$-game.

□

The fact that the minimal graph of a potential game is undirected is a simple but relevant consequence of Theorem 7.1.1. As previously mentioned, this fact can be interpreted as a first order symmetry of the interactions among players. While this pairwise symmetry is weaker than the higher order symmetry property of Theorem 7.1.1, it is significant since it can be directly read on the minimal graph of the game.

The second part of Corollary 7.1.3 is equivalent to [14, Theorems 4.2 and 4.4]. In this paper, the authors prove their results relying on the Hammersley-Clifford theorem. Our proofs are instead self-contained and in Section 7.2 we actually show that the Hammersley-Clifford theorem can be derived from our result.

In fact, we wish to emphasize that Theorem 7.1.1 is more informative than Corollary 7.1.3. Indeed, the latter does not relate the minimal separability of a potential game with that of its corresponding potential function. This is evident, e.g., in the special case of a potential game u that is pairwise separable with respect to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In this case, (79) implies that the potential function ϕ is separable with respect to the H-graph $\mathcal{H}$ where hyperlinks are the pairs $\{i, j\}$

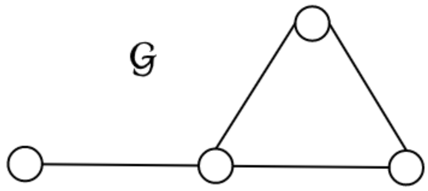


Figure 16: Graph for the pairwise separable potential game of Example 7.1.4.

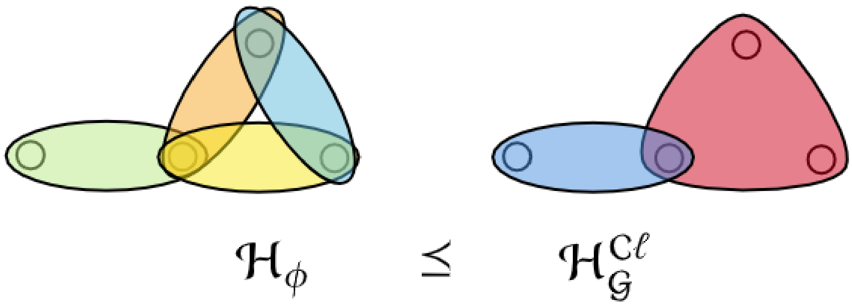


Figure 17: Comparison of $\mathcal{H}_\phi$ and $\mathcal{H}_\mathcal{G}^{\mathcal{C}\ell}$ for a pairwise separable potential game on the graph $\mathcal{G}$ of Figure 16.

such that (i, j) is a link in $\mathcal{G}$. This says that the potential can be decomposed in a pairwise fashion

$$\phi(x) = \sum_{(i,j)\in\mathcal{E}} \phi_{ij}(x_i, x_j)\,,$$

for some symmetric functions $\phi_{ij}(x_i, x_j) = \phi_{ji}(x_j, x_i)$. This is in general a much finer decomposition than the one on the maximal cliques of $\mathcal{G}$.

Example 7.1.4. *Consider a pairwise separable potential game on the graph $\mathcal{G}$ shown in Figure 16. Figure 17 compares the minimal H-graph of the potential function ϕ, $\mathcal{H}_\phi$, with the clique H-graph $\mathcal{H}_\mathcal{G}^{\mathcal{C}\ell}$, showing how in general $\mathcal{H}_\phi \preceq \mathcal{H}_\mathcal{G}^{\mathcal{C}\ell}$.*

7.2 CONNECTION WITH MARKOV RANDOM FIELDS

In this section we present in detail the connection between potential games and Markov random fields. Building on the knowledge about separable and graphical games derived from the previous Section 7.1, we enlight the connection between potential games and Markov properties for positive random fields, as introduced and discussed in Section 2.5. This analysis shows that our results on separable potential games are the counterpart of corresponding well-known results in the theory of Markov random fields. The existence of a connection between graphical potential games and Markov random fields was known [14], but in previous works structural results for potential games were proven relying on the theory of MRF. In contrast, our game theoretic results are proven independently so that we can show that there is an equivalence between our work on potential games and the foundational work on MRF. Thus, the analysis that we are presenting in this section provides a theoretical framework for our work, providing a justification for the definition of separable games. Moreover, as the theory of MRF is the foundation for recent and fruitful research, such connection with the game theoretic framework may shed light on future developments of our work on separable potential games.

7.2.1 *Positive random fields and potential games*

There is a deep connection between random fields and game theory, which is based on the equivalence between positive random vectors and potential games anticipated in Section 1.3.1.

On one side, consider a potential game u, which is characterized by a player set $\mathcal{V}$, action spaces $\{\mathcal{A}_i\}_{i\in\mathcal{V}}$, a configuration space $\mathcal{X} = \prod_{i\in\mathcal{V}} \mathcal{A}_i$ and a potential function ϕ.

To such potential game u, we can associate a positive random field $X = (X_i)_{i\in\mathcal{V}}$ where for each $i \in \mathcal{V}$, the random variable $X_i \in \mathcal{A}_i$ takes its values in the space of actions played by $i \in \mathcal{V}$, and such that the probability measure $\mathbb{P}$ over $\mathcal{X}$ satisfies

$$\mathbb{P}(X = x) = e^{\phi(x)}. \tag{83}$$

On the other side, to a positive random field $X = (X_i)_{i\in\mathcal{V}}$ we can associate a potential game u with player set $\mathcal{V}$, action spaces $\{\mathcal{A}_i\}_{i\in\mathcal{V}}$, configuration space $\mathcal{X} = \prod_{i\in\mathcal{V}} \mathcal{A}_i$, and with potential function defined by

$$\phi(x) = \log \mathbb{P}(X = x) \tag{84}$$

.

Thanks to this equivalence, we can expect that the Markov properties of random fields, and the relations among them, can be translated to the game theoretic setting to obtain analogous properties and results for potential games.

This idea will be explored in the next subsection.

7.2.2 *Markov properties and potential games*

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a positive random field $X = (X_i)_{i\in\mathcal{V}}$ with probability measure $\mathbb{P}$.

From the Hammersely-Clifford Theorem 2.5.2, we know that $\mathbb{P}$ satisfies the local Markov property (L) on $\mathcal{G}$ if and only if $\mathbb{P}$ factorizes (F) on $\mathcal{G}$.

By the equivalence that we have established, to the positive random field $\mathbb{P}$ we can associate a potential game u with potential function ϕ as in equation (84). Also, from equation (84) it directly follows that the factorization of $\mathbb{P}$ on the maximal cliques of $\mathcal{G}$ is equivalent to the decomposition of the potential on such cliques, i.e., to ϕ being separable with respect to the hypergraph of maximal cliques of $\mathcal{G}$, denoted by $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$.

Similarly, the fact that $\mathbb{P}$ satisfies the local Markov property (L) on $\mathcal{G}$ can be proven to be equivalent to the graphicality of u on $\mathcal{G}$. One side of this implication, i.e., that graphicality of u on $\mathcal{G}$ implies property (L) for $\mathbb{P}$ was proven in [14]. The remaining implication can be deduced from Corollary 7.1.3, providing the following game-theoretic derivation of the Hammersley-Clifford theorem.

Theorem 7.2.1. *Let X be a positive random field that satisfies the local Markov property (P) with respect to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then, its probability distribution admits the following decomposition:*

$$\mathbb{P}(X = x) = \prod_{\mathcal{C}\in\mathcal{C}l(\mathcal{G})} \zeta_{\mathcal{C}}(x_{\mathcal{C}}), \quad \forall x \in \mathcal{X}, \tag{85}$$

where $\mathcal{C}l(\mathcal{G})$ is the family of maximal cliques of the graph $\mathcal{G}$.

Proof. Without loss of generality we can assume that $\mathbb{P}(X_i = x_i) > 0$ for every i in $\mathcal{V}$ and x_i in $\mathcal{A}_i$ so that, by the positivity assumption we have that $\mathbb{P}(X = x) > 0$ for every x in $\mathcal{X}$. Let

$$\phi(x) = \log \mathbb{P}(X = x), \qquad \forall x \in \mathcal{X} \tag{86}$$

and consider the potential game $u = u^{\phi}$ in $\mathcal{U}$ with utility functions $u_i(x) = \phi(x)$ for every i in $\mathcal{V}$.

We shall now prove that u is $\mathcal{G}$-graphical. Indeed, conditional independence implies that

$$\begin{aligned}
\mathbb{P}(X = x) &= \mathbb{P}(X_{\mathcal{N}_i} = x_{\mathcal{N}_i})\mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i} = x_{\mathcal{V}\setminus\mathcal{N}_i} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \\
&= \mathbb{P}(X_{\mathcal{N}_i} = x_{\mathcal{N}_i})\mathbb{P}(X_i = x_i \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i})\mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} = x_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \\
&= \mathbb{P}(X_{\mathcal{N}_i^\bullet} = x_{\mathcal{N}_i^\bullet})\mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} = x_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i}),
\end{aligned}$$

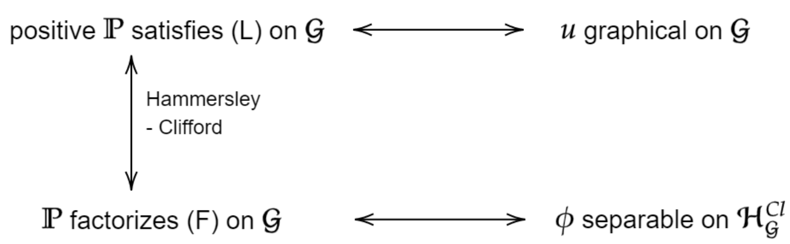


Figure 18: Graphical representation of the equivalence chain relating the local Markov property on an undirected graph $\mathcal{G}$ for a positive random field and the graphicality of the corresponding potential game.

for every i in $\mathcal{V}$. We can then write $u_i(x) = u_i^{\mathcal{N}_i}(x_i, x_{\mathcal{N}_i}) + n_i(x_{-i})$ where

$$u_i^{\mathcal{N}_i}(x_i, x_{\mathcal{N}_i}) = u_i^{\mathcal{N}_i}(x_{\mathcal{N}_i^\bullet}) = \log \mathbb{P}(X_{\mathcal{N}_i^\bullet} = x_{\mathcal{N}_i^\bullet})$$

only depends on the actions played by player i and their neighbors in $\mathcal{N}_i$, while

$$n_i(x_{-i}) = \log \mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} = x_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i})$$

is a non strategic term. Thus u is $\mathcal{G}$-graphical so that Corollary 7.1.3 implies that its potential function ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable. Together with (86), this yields the claim. □

The chain of equivalences that we have established is illustrated in Figure 18, which shows that the Hammersley-Clifford theorem of Markov random fields translates into, and is actually equivalent to, the characterization of graphical potential games proposed in [14].

As already pointed out, the result we proved, Theorem 7.1.1, is actually stronger. Indeed, we proved that if a potential game u is separable with respect to a FDH-graph $\mathcal{F}$, then its potential function ϕ is separable with respect to the H-graph $\mathcal{H}^{\mathcal{F}}$. This provides a finer decomposition of the potential. Indeed, by introducing the graph $\mathcal{G} := \mathcal{G}^{\mathcal{F}}$ on which u is graphical, we have that $\mathcal{H}^{\mathcal{F}} \subset \mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$. In other words, $\mathcal{H}^{\mathcal{F}}$ is made of cliques of $\mathcal{G}$, but not necessarily maximal cliques, so it is finer than $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$. Finally, by exploiting the relation (83), the separability of ϕ on $\mathcal{H}^{\mathcal{F}}$ implies the factorization (F) of $\mathbb{P}$ on $\mathcal{G} := \mathcal{G}^{\mathcal{F}}$. This reasoning is represented in Figure 19.

7.3 POTENTIAL-HARMONIC DECOMPOSITION OF GAMES

In this section, we consider the recent result presented in [31] regarding the decomposition of a general game into its potential and harmonic components and we investigate how the concept of separability interacts with this decomposition.

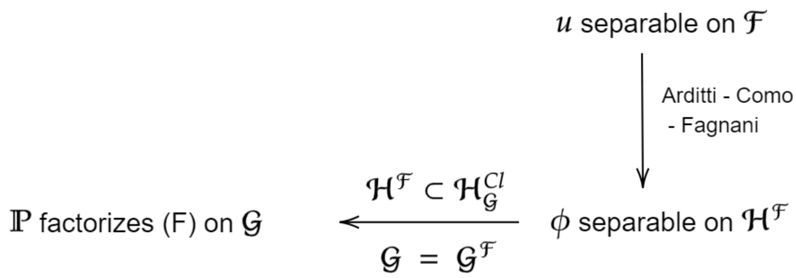


Figure 19: Representation of our Theorem 7.1.1 for a potential game u and of its implications on the factorization property of the probability distribution $\mathbb{P}$ of the positive random field associated to u.

As discussed in the background section 2.4.3, every game u can be decomposed as a sum of three games

$$u = u_{pot} + u_{har} + n \tag{87}$$

where u_{pot} is a potential game, u_{har} is a harmonic game, and n is a non-strategic game. This decomposition is unique up to non-strategic components in u_{pot} and u_{har}. In particular, it is unique if we assume that u_{pot} and u_{har} are both normalized.

The following result shows that in general the potential and harmonic components of an $\mathcal{F}$-separable game are separable on the underlying undirected FDH-graph $\mathcal{F}^{\leftrightarrow}$, whose hyperlink set can be obtained from $\mathcal{F}$'s as specified in (16).

Theorem 7.3.1. *Let u in $\mathcal{U}$ be a finite game that is $\mathcal{F}$-separable with respect to a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$. Then, u_{pot} and u_{har} are $\mathcal{F}^{\leftrightarrow}$-separable.*

Proof. We consider the decomposition (38) and for each $(i, \mathcal{J})$ in $\mathcal{D}$ we define the auxiliary game

$$u_i^{(i,\mathcal{J})}(x) = u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}), \qquad u_j^{(i,\mathcal{J})}(x) = 0, \qquad \forall j \in \mathcal{V} \setminus \{i\}. \tag{88}$$

This is a game where all players have zero utility except for player i, who has utility equal to the corresponding term $u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}})$ in (38). We can write

$$u = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u^{(i,\mathcal{J})}. \tag{89}$$

For each game $u^{(i,\mathcal{J})}$ we now consider its restriction $\hat{u}$ to the set of players $\hat{\mathcal{V}} = \{i\} \cup \mathcal{J}$ and strategy profile set $\hat{\mathcal{X}} = \mathcal{X}_{\hat{\mathcal{V}}}$. Formally,

$$\hat{u}_h(x) = \begin{cases} u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}) & \text{if } h = i \\ 0 & \text{otherwise}. \end{cases}$$

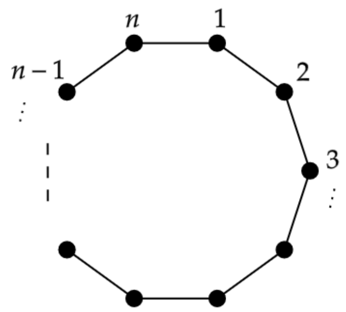


Figure 20: Ring graph of Examples 7.3.2 and 7.3.7.

Consider now the potential and the harmonic part of $\hat{u}$, respectively, $\hat{u}_{pot}$ and $\hat{u}_{har}$ and extend them to $\mathcal{V}$ as games $u_{pot}^{(i,\mathcal{J})}$ and $u_{har}^{(i,\mathcal{J})}$ such that

$$(u_{pot}^{(i,\mathcal{J})})_h = (u_{har}^{(i,\mathcal{J})})_h = 0 \quad \forall h \notin \{i\} \cup \mathcal{J}\,.$$

A direct check shows that $u_{pot}^{(i,\mathcal{J})}$ is potential, $u_{har}^{(i,\mathcal{J})}$ is harmonic and $u^{(i,\mathcal{J})} = u_{pot}^{(i,\mathcal{J})} + u_{har}^{(i,\mathcal{J})}$ up to non-strategic terms. Then, up to non strategic terms,

$$u_{pot} = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_{pot}^{(i,\mathcal{J})}, \quad u_{har} = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_{har}^{(i,\mathcal{J})}$$

Notice, finally, that since $u_{pot}^{(i,\mathcal{J})}$ and $u_{har}^{(i,\mathcal{J})}$ are separable with respect to the FDH-graph whose directed hyperlinks are of type $(h, \mathcal{K})$ with h in $\{i\} \cup \mathcal{J}$ and $\{h\} \cup \mathcal{K} = \{i\} \cup \mathcal{J}$, we have that $u_{pot}^{(i,\mathcal{J})}$ and $u_{har}^{(i,\mathcal{J})}$ are $\mathcal{F}^{\leftrightarrow}$-separable. This proves the result. □

Theorem 7.3.1 shows that the harmonic-potential decomposition of games operates on the game's separability by performing a symmetrization, which maps the FDH-graph $\mathcal{F}$ of the original game into the undirected FDH-graphs $\mathcal{F}^{\leftrightarrow}$ of the components.

We propose an example that shows how for a non-potential game u, its potential and harmonic components may display additional interdependancies among players, as captured by the underlying undirected FDH-graph $\mathcal{F}_u^{\leftrightarrow} \succeq \mathcal{F}_u$.

Example 7.3.2. *Consider the best-shot public good game as introduced in Example 3.2.2, with $n = |\mathcal{V}|$ players and defined on an undirected ring graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $n \geq 3$ nodes, as shown in Figure 20. Thanks to the symmetric structure of the ring graph, we can express the utility function of player i in $\mathcal{V}$ as*

$$u_i(x) = \max\{x_{i-1}, x_i, x_{i+1}\} - cx_i \tag{90}$$

where the algebra on the indices is intended modulo n and where $0 \leq c \leq 1$ is the cost parameter. It is clear from the form of the utility in (90) that the game u is separable on the FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ with set of directed hyperlinks

$$\mathcal{D} = \{(i, \{i-1, i+1\}),\, i \in \mathcal{V}\} \ .$$

In fact, since it is not possible to rewrite the max *term in (90) as the sum of three terms depending on (x_{i-1}, x_i), (x_i, x_{i+1}), and (x_{i-1}, x_{i+1}), respectively, the above is the minimal FDH-graph of u (as shown in Example 4.1.6). The utility functions of the normalized potential and harmonic components are given by*

$$\begin{aligned} u_{pot,i}(x) = (&|x_{i+1} - x_{i+2}| + |x_{i-1} - x_{i-2}| + 4(x_{i+1} + x_{i-1}) \\ &-2x_{i+1}x_{i-1} - 6(1-c))\,\frac{1-2x_i}{12}\,, \end{aligned} \tag{91}$$

and, respectively,

$$u_{har,i}(x) = (|x_{i+1} - x_{i+2}| + |x_{i-1} - x_{i-2}| - 2(x_{i+1} + x_{i-1}))\,\frac{1-2x_i}{12}\,, \tag{92}$$

for every player i in $\mathcal{V}$. Notice that the harmonic component is independent from c, while the potential component contains an additive term that is linear on c and that depends only on the action of player i itself. According to the decomposition result of Theorem 7.3.1, equations (91) and (92) show that the normalized potential and harmonic components of u are separable on the FDH-graph $\mathcal{F}^{\leftrightarrow} = (\mathcal{V}, \mathcal{D}^{\leftrightarrow})$ where

$$\mathcal{D}^{\leftrightarrow} = \{(i, \{i-2, i-1\}), (i, \{i-1, i+1\}), (i, \{i+1, i+2\}),\ i \in \mathcal{V}\} \ .$$

Accordingly, the potential function ϕ of u_{pot} is separable on the H-graph $\mathcal{H}^{\mathcal{F}^{\leftrightarrow}} = \mathcal{H}^{\mathcal{F}} = (\mathcal{V}, \mathcal{L})$ with set of undirected hyperlinks

$$\mathcal{L} = \{\{i-1, i, i+1\},\, i \in \mathcal{V}\} \ ,$$

that is displayed in Figure 21.

Theorem 7.3.1 has a few important direct consequences that we discuss in the following. First, we can transfer the result into the setting of graphical games. This is obtained by translating the symmetrization operation from FDH-graphs to the corresponding graphs as illustrated in Figure 22. A preliminary version of the following result was presented in our previous work [8].

Corollary 7.3.3. *Let u in $\mathcal{U}$ be a finite game that is $\mathcal{G}$-graphical with respect to a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then, u_{pot} and u_{har} are $\mathcal{G}^{\triangle}$-graphical.*

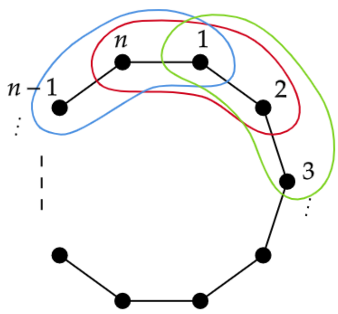


Figure 21: Hypergraph of the potential ϕ for Example 7.3.2.

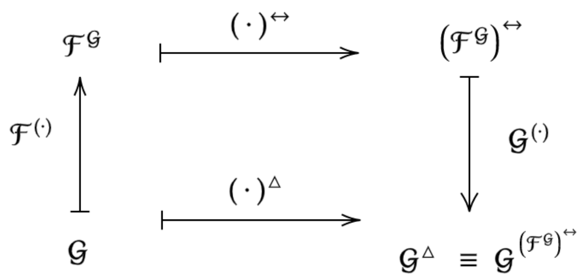


Figure 22: Transposition of the symmetrization operation from FDH-graphs to graphs.

Proof. The statement follows from Theorem 7.3.1 upon proving that the diagram in Figure 22 commutes, i.e., that

$$\mathcal{G}^{(\mathcal{F}^{\mathcal{G}})^{\leftrightarrow}} = \mathcal{G}^{\triangle}. \tag{93}$$

To simplify the notation, we denote $\mathcal{G}^* = \mathcal{G}^{(\mathcal{F}^{\mathcal{G}})^{\leftrightarrow}}$ and by $\mathcal{N}_i^*$ the neighborhood of player $i \in \mathcal{V}$ in $\mathcal{G}^*$. Moreover, we denote by $\mathcal{N}_i^{\triangle}$ the neighborhood of player $i \in \mathcal{V}$ in $\mathcal{G}^{\triangle} = (\mathcal{V}, \mathcal{E}^{\triangle})$ as defined in (8). Equation (93) follows from the following chain:

$$\begin{aligned}
\mathcal{N}_i^* &= \bigcup \left\{ \mathcal{J} : (i, \mathcal{J}) \in \mathcal{D}((\mathcal{F}^{\mathcal{G}})^{\leftrightarrow}) \right\} \\
&= \bigcup \left\{ \mathcal{J} : \{i\} \cup \mathcal{J} = \{h\} \cup \mathcal{K}, (h, \mathcal{K}) \in \mathcal{D}(\mathcal{F}^{\mathcal{G}}), \mathcal{J} \subset \mathcal{V} \setminus \{i\} \right\} \\
&= \bigcup \{ \mathcal{J} : \{i\} \cup \mathcal{J} = \mathcal{N}_h^{\bullet}, \mathcal{J} \subset \mathcal{V} \setminus \{i\}, h \in \mathcal{V} \} \\
&= \bigcup \{ \mathcal{N}_h^{\bullet} \setminus \{i\} : h \in \mathcal{V}, i \in \mathcal{N}_h^{\bullet} \} \\
&\equiv \mathcal{N}_i^{\triangle},
\end{aligned}$$

where to ease the notation we denote with $\mathcal{D}(\cdot)$ the set of hyperedges of an FDH-graph. □

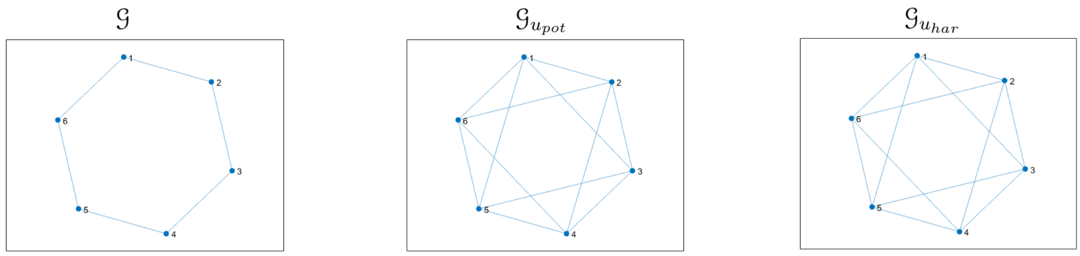


Figure 23: Representation of the components' minimal graphs $\mathcal{G}_{u_{pot}}$ and $\mathcal{G}_{u_{har}}$ for a public good game on $\mathcal{G}$, as discussed in Example 7.3.4, obtained with the procedure reported at https://github.com/laura-arditti/game-decomposition.

Corollary 7.3.3 states that the minimal graphs $\mathcal{G}_{u_{pot}}$ and $\mathcal{G}_{u_{har}}$ of the potential and, respectively, harmonic components of a game u are subgraphs of $\mathcal{G}^{\triangle}$. As shown in Figure 2b in general $\mathcal{G}^{\triangle}$ is a strict supergraph of $\mathcal{G}$, which suggests that the nonstrategic-harmonic-potential decomposition of graphical games may not preserve graphicality. In fact, Corollary 7.3.3 allows for the possibility that the minimal graphs of the potential and harmonic components of a finite game u include links between two players j and k that are not direct neighbors in $\mathcal{G}$ but share a common in-neighbor i. While not influencing directly their respective utilities, such players j and k both directly influence the utility of player i and this may result in the appearance of a link between them in the minimal graphs of the normalized potential and normalized harmonic components of the game, as shown by the following example.

Example 7.3.4 (Decomposition of the public good game)**.** *Consider the best-shot public good game defined in Example 3.2.2 over a cycle graph with* 6 *nodes.*

Figure 23 shows the minimal graph $\mathcal{G}$ *of the game and the minimal graphs associated to its potential and harmonic components. These are obtained with an explicit computation performed via the program published at* https://github.com/laura-arditti/game-decomposition*, which combines the flow description of the harmonic-potential decomposition (presented in Sections 2.4.2 and 2.4.3) with the geometric characterization of graphicality derived in Chapter 8. According to Corollary 7.3.3, they are both subgraphs of* $\mathcal{G}^{\triangle}$*. In this particular case they coincide with* $\mathcal{G}^{\triangle}$*, showing that a sharper result cannot be obtained.*

On the other hand, such possible links between common outneighbors of a single player in $\mathcal{G}$ may not show up in the minimal graphs $\mathcal{G}_{u_{pot}}$ and $\mathcal{G}_{u_{har}}$ of the potential and normalized components of the game. In fact, in the special case of pairwise separable games Theorem 7.3.1 reduces to the following.

Corollary 7.3.5. *Let u in $\mathcal{U}$ be a pairwise separable game on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with utilities as in (34). Then u_{pot} and u_{har} are pairwise separable games on $\mathcal{G}^{\leftrightarrow}$.*

Proof. As discussed in Example 3.2.1 for u to be pairwise separable on $\mathcal{G}$ it means that u is separable with respect to the FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ where

$$\mathcal{D} = \{(i, \{j\}), i \in \mathcal{V}, j \in \mathcal{N}_i\}.$$

Then, Theorem 7.3.1 guarantees that u_{pot} and u_{har} are separable on $\mathcal{F}^{\leftrightarrow}$, which, by the same token, assures that they are pairwise separable on $\mathcal{G}^{\leftrightarrow}$. □

Corollary 7.3.5 shows that for pairwise separable games the decomposition (7.3) preserves the original graphical structure in that no link between players that were not directly interacting in the original game shows up in either $\mathcal{G}_{u_{pot}}$ or $\mathcal{G}_{u_{har}}$.

In fact, when $\mathcal{G}$ is undirected, we have that $\mathcal{G}^{\leftrightarrow}$ is a subgraph of $\mathcal{G}$ so that we can simplify the result to deduce that u_{pot} and u_{har} are graphical with respect to $\mathcal{G}$.

Example 7.3.6 (Decomposition of pairwise separable games)**.** *Consider an undirected graph $\mathcal{G}$ and construct a pairwise separable game where each pair of adjacent players is involved either in a coordination, an anticoordination or a discoordination game, as in Examples 2.4.8, 2.4.6 and 2.4.1. Denote by $\mathcal{C}$, $\mathcal{A}$, $\mathcal{D}$ the sets of pairs of players involved in a coordination, anticoordination or discoordination game, respectively.*

If we perform the decomposition according to Corollary 7.3.5, we obtain that

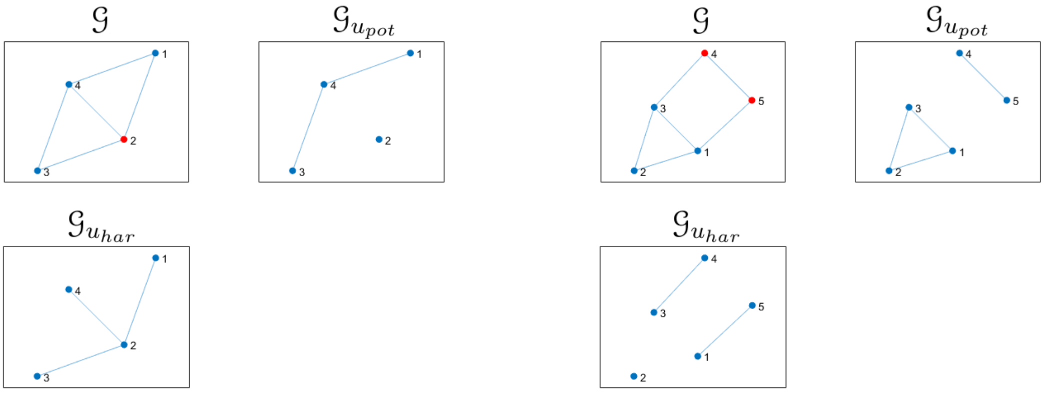
- *$\mathcal{G}_{u_{pot}}$ contains the link $\{i, j\} \in \mathcal{E}$ if and only if $\{i, j\} \in \mathcal{C}$ or $\{i, j\} \in \mathcal{A}$, i.e. if i and j are involved in either a coordination or an anticoordination game;*
- *$\mathcal{G}_{u_{har}}$ contains the link $\{i, j\} \in \mathcal{E}$ if and only if $\{i, j\} \in \mathcal{D}$, i.e. if i and j are involved in a discordination game.*

So in this case the graphs $\mathcal{G}_{u_{pot}}$ and $\mathcal{G}_{u_{har}}$ can be determined by inspection and their link sets are disjoint. In Table 24 we report some examples of decomposition. In particular we consider two topologies for $\mathcal{G}$. Red (blue) nodes indicate players who anticoordinate (coordinate) with their neighbors.

As discussed, by Corollary 7.3.5, when u is pairwise-separable its potential and harmonic components are also pairwise separable. We notice however that this condition is only sufficient, as the following example shows.

Example 7.3.7. *For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consider the game u with player set $\mathcal{V}$, action set $\mathcal{A}_i = \{0, 1\}$, and utilities*

$$u_i(x) = \begin{cases} 1 & \text{if } x_i = x_j, \forall j \in \mathcal{N}_i \\ 0 & \text{otherwise}, \end{cases} \tag{94}$$



(a) Case I (b) Case II

Figure 24: Representation of $\mathcal{G}_{u_{pot}}$ and $\mathcal{G}_{u_{har}}$ for pairwise-separable graphical game on two different graphs $\mathcal{G}$. The minimal graphs of the game's components are obtained and plotted with the procedure reported at `https://github.com/laura-arditti/game-decomposition`.

Figure 25: Line graph L_n for Example 7.3.7.

for every player i in $\mathcal{V}$. Notice that, for a general graph $\mathcal{G}$, u is neither a potential game nor pairwise separable on $\mathcal{G}$. We shall now study the structure of the game u for two different graph topologies.

When $\mathcal{G}$ is an undirected ring graph with n nodes, as in Figure 20, then u is a potential game and it is strategically equivalent to a multiple of the network coordination game on $\mathcal{G}$ introduced in Example 2.4.8 with $\zeta(x_i, x_j) = (-1)^{x_i - x_j}$, where the pairwise utilities are scaled by $\frac{1}{4}$. So, the normalized potential component u_{pot} is a network coordination game on $\mathcal{G}$, and it is pairwise separable on $\mathcal{G}$.

When $\mathcal{G} = L_n$ is an undirected line graph with n nodes, as displayed in Figure 25, then u is not a potential game. The normalized potential component u_{pot} is a weighted version of the network coordination game on the line L_n, as represented in Figure 26, where the pairwise utility function $\zeta(x_i, x_j) = (-1)^{x_i - x_j}$ is multiplied by different factors depending on the link $\{i, j\}$.

In particular,

- *for the internal links, joining nodes from 2 to $n-1$, the pairwise utility ζ is scaled by $\frac{1}{4}$,*

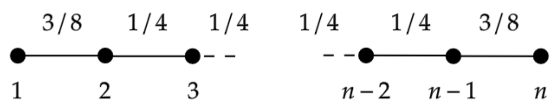


Figure 26: Line graph for the potential component of Example 7.3.7. For each link $\{i, j\}$ the factor multiplying the pairwise utility $\zeta(x_i, x_j)$ is specified.

Figure 27: Graph for the harmonic component of Example 7.3.7.

- *for the extremal links $\{1, 2\}$ and $\{n-1, n\}$, the pairwise utility ζ is scaled by $\frac{3}{8}$.*

The harmonic component is pairwise separable on a graph with nodes $\mathcal{V}$ and with only two undirected links, namely $\{1, 2\}$ and $\{n-1, n\}$, which is shown in Figure 27. On such links two so-called matching pennies game take place, where nodes 1 and n aim to coordinate with their neighbors, while 2 and $n-1$ aim to anti-coordinate.

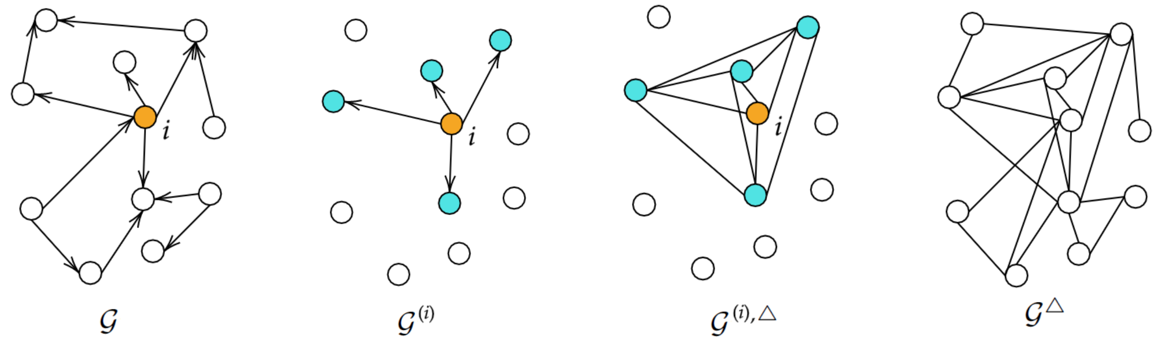
In conclusion, Theorem 7.3.1 provides a characterization of the minimal FDH-graph $\mathcal{F}^{\leftrightarrow}$ of the potential and harmonic components of an $\mathcal{F}$-separable game u. When reflected on the graphicality of the game's components, it states that the graphs $\mathcal{G}_{u_{pot}}$ and $\mathcal{G}_{u_{har}}$ are contained in $\mathcal{G}^{\mathcal{F}^{\leftrightarrow}}$, which is in general a subgraph of $(\mathcal{G}^{\mathcal{F}})^{\triangle}$ and a supergraph of $(\mathcal{G}^{\mathcal{F}})^{\leftrightarrow}$, the two extreme cases being realized for the pairwise separable and the generic graphical case.

7.3.1 *Locality of the game decomposition*

The main takeaway from the proof of Theorem 7.3.1 and Corollary 7.3.3 is that the harmonic-potential game decomposition can be performed locally.

To clarify this point, in Figure 28 we visually retrace the sketch of the proof for the case of graphical games, as it follows the same lines of the most general case of separable games while allowing for a simpler visual representation.

We start with a graphical game u over $\mathcal{G}$, depicted in Figure (28,a), which can be represented as a vector $u = (u_1, \ldots, u_n)$ collecting the utility functions of the n players. The game u can be expressed as the sum $u = u^{(1)} + \ldots + u^{(n)}$ of n games, each associated to one player: the game associated to player i, $u^{(i)}$, is a game where all players have vanishing utility except for i that has utility $u_i^{(i)} = u_i$ equal to the one she had in u. The local game $u^{(i)}$ is graphical on the subgraph $\mathcal{G}^{(i)} = (\mathcal{V}, \mathcal{E}^{(i)})$ of the original graph $\mathcal{G}$ where $\mathcal{E}^{(i)} = \{(i, j) : j \in \mathcal{N}_i\}$, shown in Figure (28,b), that is a star graph with center i and leaf nodes corresponding to the



(a) Graph $\mathcal{G}$ of the game u.

(b) Graph $\mathcal{G}^{(i)}$ of the local game $u^{(i)}$.

(c) Graph $\mathcal{G}^{(i),\triangle}$ of $u^{(i)}_{pot}$ and $u^{(i)}_{har}$.

(d) Graph $\mathcal{G}^{\triangle}$ of u_{pot} and u_{har}.

Figure 28: Sketch of the proof of Corollary 7.3.3.

neighbours $\mathcal{N}_i$ of i in $\mathcal{G}$. When decomposing the local game $u^{(i)}$, its potential and harmonic components $u^{(i)}_{pot}$ and $u^{(i)}_{har}$ are graphical on the triangulation $\mathcal{G}^{(i),\triangle}$ of the local star graph shown in Figure (28,c), which is a complete graph on the closed out-neighoubrhood $\mathcal{N}_i$ of i. By linearity, the potential and harmonic component of the starting game u can be obtained by summing the potential and harmonic components of all local games $u^{(i)}$: in this way we obtain that the components of u are graphical on the triangulation $\mathcal{G}^{\triangle}$ of $\mathcal{G}$, represented in Figure (28,d), which is the union of all $\mathcal{G}^{(i),\triangle}$ associated to each player.

In conclusion, the key point of the proof is that the decomposition can be performed locally, neighborhood by neighborhood (or, more in general, hyperlink by hyperlink). This locality property of the decomposition have some further implications, as can be seen from the following example.

Example 7.3.8. *Consider the graph $\mathcal{G}$ represented in Figure 29 that describes the interactions between 8 players. We define a $\mathcal{G}$-game where all players have a binary action set $\mathcal{A} = \{0, 1\}$. Action 1 and 0 represent the action of acquiring or not acquiring some good, and we assume that all players but player 1 have an imitative behaviour so that their behaviour is described by the utility function of a majority game (see Example 2.4.8):*

$$\forall i \in \mathcal{V} \setminus \{1\}, \quad u_i(x) = |\{j \in \mathcal{N}_i : x_j = x_i\}|, \qquad x_i \in \{0, 1\}.$$

Instead, player 1, represented by a red node, plays according to the utility function of a best shot public good game (see Example 3.2.2), i.e., for some $0 < c < 1$ it holds:

$$u_1(x) = \begin{cases} 1 - c \text{ if } x_1 = 1 \\ 1 \text{ if } x_1 = 0 \text{ and } x_j = 1 \text{ for some } j \in \mathcal{N}_1 \\ 0 \text{ if } x_1 = 0 \text{ and } x_j = 0 \text{ for every } j \in \mathcal{N}_1 \,. \end{cases}$$

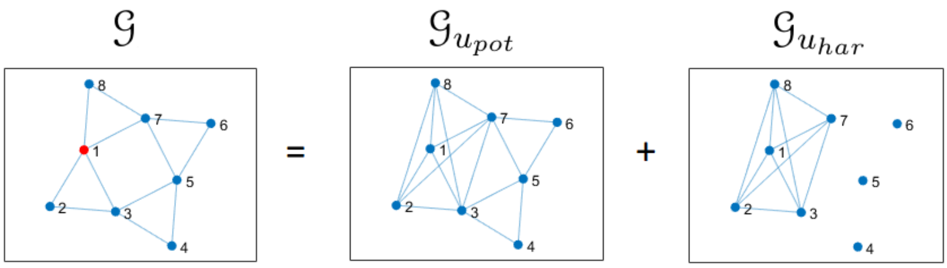


Figure 29: Graphicality of the components for the decomposition of the perturbed majority game of Example 7.3.8.

The resulting game is not potential but it is a local perturbation of a potential game, the majority game, where the perturbation is localized in the red node. In Figure 29 we see that the locality of the perturbation is preserved by the decomposition. Indeed, players that are far from the perturbation only interact in the graph of the potential component while additional links with respect to the original graph only involve players that are close to the perturbation and directly affected by it. This phenomenon is the result of the combination of locality of the decomposition and of the perturbation.

The perturbative approach taken in Example 7.3.8 can be applied more in general to study properties of graphical games, such as Nash equilibria. We briefly comment on this line of research, showing how our characterization of the graphicality of the game decomposition may be exploited in this setting, leaving the details for future investigations.

Consider the problem of finding Nash equilibria of a $\mathcal{G}$-game u. In view of the potential-harmonic decomposition of games, the harmonic part u_{har} can be interpreted as a perturbation of the potential component u_{pot}. Upon formally defining some notions of robustness of a Nash equilibria and of magnitude of perturbations, we have that if u_{pot} possess a sufficiently robust Nash equilibrium x^* and if perturbation u_{har} is small enough, then x^* is a Nash equilibrium of the whole game u. One possible way of defining the robustness of a Nash equilibrium x is by

$$\min_{i \in \mathcal{V}} \min_{\substack{a \in \mathcal{A}_i \\ a \neq x_i}} u_i(x) - u_i(a, x_{-i}),$$

while one possible way of measuring the magnitude of a perturbation u_{har} is via its L^∞-norm

$$\max_{i \in \mathcal{V}} \max_{x \in \mathcal{X}} |u_i(x)|.$$

We don't dive here into a description of such definitions, instead we wish to point out how the graphical representation of u can be exploited to bound the complex-

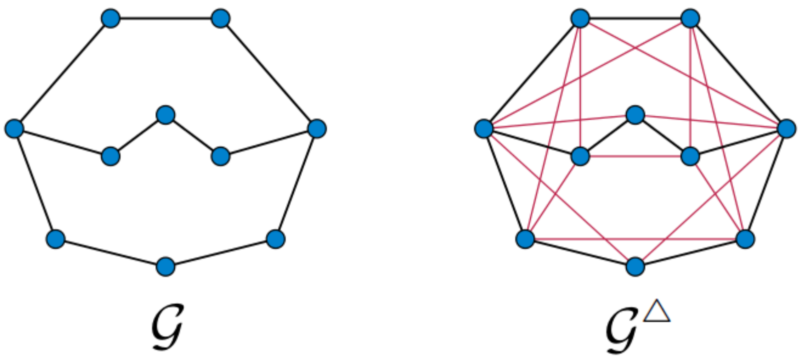


Figure 30: The size of local games for potential and harmonic components of a $\mathcal{G}$-game are bounded by the size of neighborhoods of $\mathcal{G}^{\triangle}$ (picture from https://en.wikipedia.org/wiki/Graph_power).

ity of computing these quantities. Indeed, the robustness of a Nash equilibrium of u_{pot} and the magnitude of the perturbation u_{har} can be computed on the local games $u^{(i)}_{pot}$ and $u^{(i)}_{har}$, whose size is bounded by the size of neighborhoods of $\mathcal{G}^{\triangle}$. Usually (see Figure 30), $\mathcal{G}^{\triangle}$ is much smaller than the complete graph on $\mathcal{V}$, possibly resulting in an efficient implementation of the perturbative analysis of games outlined above.

7.3.2 *Application to an economic model*

In this section we present a simple example of how the harmonic-potential decomposition and our results about the structure of components can be applied to games emerging from real world models and we show how they are able to highlight relevant features of the system. For simplicity, we limit our discussion to the graphical case but similar considerations hold more in general for separable games.

Consider a set of players $\mathcal{V} = \mathcal{V}_p \cup \mathcal{V}_c$ where the elements in $\mathcal{V}_p$ represent producers of some good while the elements in $\mathcal{V}_c$ represent consumers. We denote the size of those sets with $n_p = |\mathcal{V}_p|$ and $n_c = |\mathcal{V}_c|$. Players in $\mathcal{V}$ are involved in the game u, which we call "market game", described by the following specifications.

- Each producer $i \in \mathcal{V}_p$ choses the quality level $a_i \in \mathcal{A}^p$ of its product from a finite set. For simplicity we will consider binary sets $\mathcal{A}^p$, i.e., the producers chose to produce either an higher or a lower quality product. This choice will be reflected on both the production cost (for the producer) of the product and its selling price (for the consumers), which will be higher in the former and lower in the latter case respectively.

- Each consumer $j \in \mathcal{V}_c$ chooses from which producer $a_j = i \in \mathcal{V}_p \equiv \mathcal{A}^c$ they will buy a unit of product.

Denote with

$$d(x,i) = |\{j \in \mathcal{V}_c : x_j = i\}|$$

the number of consumers that choose producer i in configuration $x \in \mathcal{X} = \Pi_{i \in \mathcal{V}_p} \mathcal{A}^p \times \Pi_{j \in \mathcal{V}_c} \mathcal{A}^c$. Let $x \in \mathcal{X}$ be a configuration.

- The utility of producer $i \in \mathcal{V}_p$ is

  $$u_i(x) = p_i(x_i) d(x,i) - c_i(x_i, d(x,i)) \tag{95}$$

  where p_i is the selling price of one unit of the product, determined by its quality, and c_i is the total cost of production for i, which depends on the quality of the product and on the total produced quantity (which we assume equal to the demand $d(\cdot, i)$).

- The utility of consumer $j \in \mathcal{V}_c$ is

  $$u_j(x) = -\sum_{i \in \mathcal{V}_p} \delta(i, x_j) p_i(x_i) \tag{96}$$

  where

  $$\delta(i, x_j) = \begin{cases} 1 & \text{if } i = x_j \\ 0 & \text{otherwise} \end{cases}$$

  i.e., it is equal to the opposite of the price they pay to get a unit of product from the producer they chose. Notice that in this formulation of the game, consumers are not sensitive to quality but only to price.

Consumers observe the whole market when taking their choice and their utility depends on the prices set by all producers. On the other side, producers observe the whole set of consumers and their utility depends on the choice of all of them. Moreover, both consumers and producers are not affected directly by the behaviour of other consumers and producers respectively. As a consequence, interactions are described by a complete bipartite graph as in Figure 31.

Observe that in general the game is not pairwise separable, since the cost functions of producers are not necessarily linear. A significant assumption is that cost functions are concave with respect to demand. With this assumption the marginal cost of producing a unit of product is decreasing in the total produced quantity. This is the case of an economy of scale. According to this, we consider that the production cost of producer i is a fraction $\frac{f_i(d(x,i))}{100}$ of its return $p_i(x_i)d(x,i)$, which

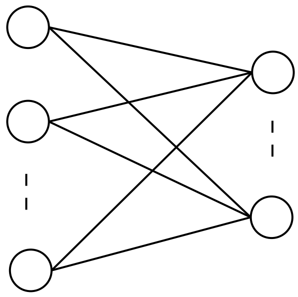


Figure 31: Interaction graph for the market game

decrease as the total produced quantity grows. In this way, the cost grows when the quality (and then the price) of the product increase. Then the utility of producer $i \in \mathcal{V}_p$ takes the following form:

$$u_i(x) = p_i(x_i)d(x,i)\frac{(100 - f_i(d(x,i)))}{100} \tag{97}$$

We focus on the case of $n_p = n_c = 2$, when there are two consumers and two producers. Then we also have that $|\mathcal{A}^c| = |\mathcal{A}^p| = 2$. In this setting we analyse an instance of the market game, we perform the harmonic-potential decomposition of it and we describe the insights that can be derived.

Figure 32 shows the minimal graph of the game, a $(2,2)$-complete bipartite graph, and the minimal graphs of its potential and harmonic components.

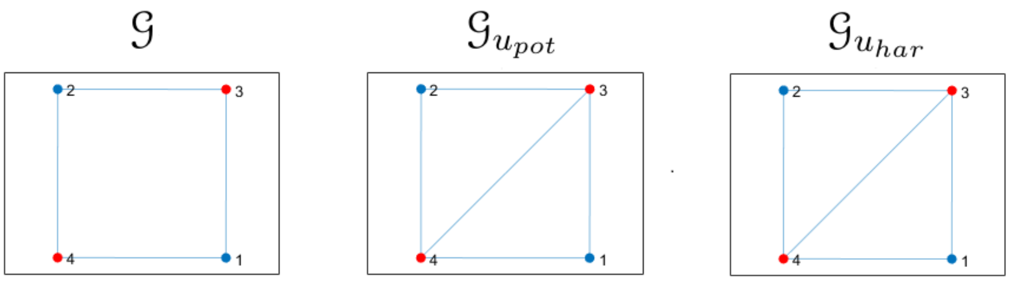


Figure 32: Harmonic-potential decomposition of the market game. Red nodes represent consumers, while blue nodes represent producers.

We see that the market game possesses a potential component where a direct relation between consumers appears, while this is not the case for the producers.

This fact can be explained as follows. Call $u^{(i)}$, $i = 1, \dots, n_p + n_c$ the local games involving only players $\{i\} \cup \mathcal{N}_i$ where i has utility $u_i^{(i)} = u_i$ while every player in $\mathcal{N}_i$ has vanishing utility (as introduced and discussed in Section 7.3.1). An edge between the producers (players 1 and 2) in $\mathcal{G}_{pot}$ and $\mathcal{G}_{har}$ can appear only if it is present in the decomposition of the restricted games $u^{(3)}$ or $u^{(4)}$ associated to consumers. But these two games are pairwise separable games on star graphs centered at nodes 3 and 4 respectively and with two leaves, namely nodes 1 and 2. As by Corollary 7.3.5, the decomposition of pairwise separable games preserves graphicality, so that no edge between nodes 1 and 2 can appear.

This is just a formal motivation but it is possible to give a more intuitive one. The potential ϕ of the potential component u_{pot} represents a quantity that the potential component of the system aims at maximizing. The potential component of the system benefits if consumers coordinate their actions and choose to buy from the same producer, since the total costs of production are lower if they do. This happens since we have modelled an economy of scale: in the opposite case (cost functions which are convex with respect to the total demand) the system benefits if consumers buy from different producers. The decomposition highlights these strategic considerations which are not evident by simply looking at the game since they are hidden in indirect interactions among players.

7.4 BETTER-RESPONSE PATHS IN SEPARABLE POTENTIAL GAMES

Separability has some direct implications on the analysis of better response paths of potential games. These are sequences of configurations $(x_t)_t$ such that each couple of consecutive configurations (x_{k-1}, x_k) differ exactly for the action of a single player, say $i \in \mathcal{V}$, which is strictly increasing their utility $u_i(x_k) > u_i(x_{k-1})$. A relevant problem in this setting is whether the length of BR paths can be bounded (non-trivially), as this has major consequences on the game's evolution under relevant dynamics. In the context of graphical games, this problem has been studied in [14], where [14, Theorem 5.3] provides a bound on the number of any player's k updates in better response paths of a graphical potential game with integer potential. A crucial role in this result is played by the *structure* of the potential function, in particular its decomposition on the graph's cliques, and by the *growth* of the graph. The latter is measured by means of the quantity $S_r(\mathcal{G}, k) = |\{j : \Delta(j,k) = r\}|$, which is the number of nodes at distance r from node k in the graph $\mathcal{G}$, and it affects the results in that the proposed bound is remarkably independent on the number of players for graphs of sufficiently slow growth. Instead, the structure of the potential function comes into play both through the size of local potentials associated to cliques and by the way cliques are arranged over the graph. These two features are combined into a single quantity $p = \max_{i \in \mathcal{V}} D_i M_i$, which is the maximum product of the largest magnitude of *clique local potentials* over cliques containing player i, $M_i = \max_{\mathcal{C} \in \mathcal{C}\ell(\mathcal{G}) : i \in \mathcal{C}} \max_{x \in \mathcal{X}}, \phi_{\mathcal{C}}(x_{\mathcal{C}})$, and of the *clique degree* of player i, $D_i = |\{\mathcal{C} \in \mathcal{C}\ell(\mathcal{G}) : i \in \mathcal{C}\}|$. More precisely, [14, Theorem

5.3] bounds the number of updates of any player $k \in \mathcal{V}$ in any better response path by

$$2p \sum_{r=0}^{\infty} \left(1 - \frac{1}{2p}\right)^r S_r(\mathcal{G}, k). \tag{98}$$

In the context of separable games we can adapt this result by exploiting the finer decomposition of the potential function proved in Theorem 7.1.1. In particular, for an $\mathcal{F}$-separable potential game we can give a bound analogous to (98) where p is now defined as the maximum product of the players' *hyper-degree* in $\mathcal{H}^{\mathcal{F}} = (\mathcal{V}, \mathcal{L})$

$$D_i = |\{\mathcal{K} \in \mathcal{L} : i \in \mathcal{K}\}| \tag{99}$$

and the maximum magnitude of *hyperlink local potentials*

$$M_i = \max_{\substack{\mathcal{K} \in \mathcal{L} \\ \text{s.t. } i \in \mathcal{K}}} \max_{x \in \mathcal{X}} |\phi_{\mathcal{K}}(x_{\mathcal{K}})|. \tag{100}$$

Our result can be stated as follows.

Theorem 7.4.1. *Let u be a potential game with potential ϕ, separable on the H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$. Assume that all local potentials are integer and define p as the maximum product of the two quantities (99) and (100):*

$$p = \max_i D_i M_i. \tag{101}$$

Then, the number of times a player $k \in \mathcal{V}$ updates their action in a better response path for u is at most

$$2p \sum_{r=0}^{\infty} \left(1 - \frac{1}{2p}\right)^r S_r(\mathcal{G}, k), \tag{102}$$

where $\mathcal{G} = \mathcal{G}^{\mathcal{F}^{\mathcal{H}}}$.

Proof. The proof follows the same lines of the proof of [14, Theorem 5.3] (which is reported in [14, Section 5.1]). We present it in the Appendix 7.4.2 at the end of this section for completeness, adapting it to our setting. □

Theorem 7.4.1 is particularly relevant in that, when the growth of the graph $\mathcal{G}^{\mathcal{F}}$ expressed by $S_r(\mathcal{G}, k)$ is sufficiently slow, it allows obtaining bounds on the number of players' updates in any better response path that are independent on the size of the graph, i.e., on the number of players.

7.4.1 *The role of separability*

To understand the role of separability we compare our result with the analogous [14, Theorem 5.3] obtained within the graphical game setting.

First, we notice that Theorem 7.4.1 is an extention of [14, Theorem 5.3]. Indeed, for a graphical potential game u with minimal graph $\mathcal{G}$ and potential ϕ, according to Corollary 7.1.3 it holds that

$$\mathcal{H}_\phi \preceq \mathcal{H}_\mathcal{G}^{\mathcal{C}l}.$$

We can then apply Theorem 7.4.1 to u exploiting the H-graph $\mathcal{H}_\mathcal{G}^{\mathcal{C}l}$, obtaining the same result of [14, Theorem 5.3]. Indeed, in this case the clique-degree and the maximum magnitude of local clique potentials over $\mathcal{G}$ coincide with D_i and M_i as defined in (99) and (100) respectively, since H-degrees in $\mathcal{H}_\mathcal{G}^{\mathcal{C}l}$ coincide with clique degrees in $\mathcal{G}$ and the decompositions of the potential with respect to the H-graph $\mathcal{H}_\mathcal{G}^{\mathcal{C}l}$ or the graph $\mathcal{G}$ are the same. As a consequence the upper bound obtained using Theorem 7.4.1 with $\mathcal{H} = \mathcal{H}_\mathcal{G}^{\mathcal{C}l}$ is as low as the one from [14, Theorem 5.3].

Actually, taking advantage of a game's separability may result in an improvement on [14, Theorem 5.3]. The key idea is that for an $\mathcal{F}$-separable potential game the decomposition of the potential over hyperlinks of $\mathcal{H}^\mathcal{F}$ is in general finer than the one on maximal cliques of $\mathcal{G}^\mathcal{F}$, so that the hyper-degree of each player in $\mathcal{H}^\mathcal{F}$ will be larger than its clique-degree in $\mathcal{G}^\mathcal{F}$ while local potentials over hyperlinks of $\mathcal{H}^\mathcal{F}$ will have smaller magnitude than clique potentials of $\mathcal{G}^\mathcal{F}$. The combination of this two effects may result, for a separable potential game, in a reduction of the parameter p with respect to its graphical representation and, consequently, an improvement of the bound obtained in [14, Theorem 5.3].

To illustrate this reasoning we propose an example showing that the knowledge of the minimal separability of the potential function ϕ can be exploited to obtain a better upper bound on the number of players' updates in better response paths.

Example 7.4.2. *Consider an augmented grid graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $n = |\mathcal{V}|$ nodes, as represented in Figure 33 and define a partition of the link set as $\mathcal{E} = \mathcal{E}^C + \mathcal{E}^A$ in such a way that the two subgraphs $\mathcal{G}^A = (\mathcal{V}, \mathcal{E}^A)$ and $\mathcal{G}^C = (\mathcal{V}, \mathcal{E}^C)$ are undirected, i.e., such that $(i, j) \in \mathcal{E}^C \Leftrightarrow (j, i) \in \mathcal{E}^C$.*

Observe that the augmented grid graph with n nodes has linear growth, since for any $k \in \mathcal{V}$, $\mathcal{S}_0(\mathcal{G}, k) = 1$ and $\mathcal{S}_r(\mathcal{G}, k) \leq 8r$ for all $r > 0$. This fact implies that any bound obtained with Theorem 7.4.1 converges to a constant value, which is independent on the number of players, as the size of the graph grows sufficiently large.

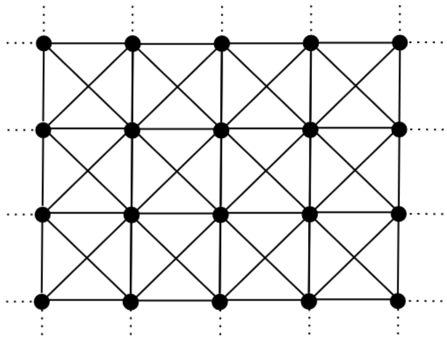


Figure 33: A portion of the augmented grid graph considered in Example 7.4.2.

Consider a pairwise separable game on $\mathcal{G}$ with binary actions $\mathcal{A}_i = \{0, 1\}$ for all $i \in \mathcal{V}$ and utility functions

$$u_i(x) = \sum_{\substack{j\in\mathcal{V} \\ (i,j)\in\mathcal{E}^C}} 1 - |x_i - x_j| - \sum_{\substack{j\in\mathcal{V} \\ (i,j)\in\mathcal{E}^A}} 1 - |x_i - x_j|.$$

Such game, which is also discussed in [14, Example 5.4], can be interpreted as a mixed coordination/anti-coordination game where each player simultaneously coordinate and anti-coordinate with different neighbours depending on the type of their connection, described by the partition of the link set. The game is potential with potential function given by

$$\phi(x) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}^C} 1 - |x_i - x_j| + \frac{1}{2} \sum_{(i,j)\in\mathcal{E}^A} |x_i - x_j|. \tag{103}$$

Consider the minimal H-graph of the potential, $\mathcal{H} = \mathcal{H}_\phi = (\mathcal{V}, \mathcal{L})$. Since the game is pairwise separable, we have that $\mathcal{L} = \{\{i, j\} \in \mathcal{E}\}$. Moreover, for all $i \in \mathcal{V}$, $D_i(\mathcal{H}) = 8$ while $M_i(\mathcal{H}) = 1$, since the potential ϕ can be decomposed as a sum of local potentials $\phi_\mathcal{J}$ on links, which are all bounded by 1. As a consequence, we have that $p(\mathcal{H}) = 8$. Then, the corresponding bound on the number of any player k's updates in any better response path obtained from Theorem 7.4.1 is

$$2p(\mathcal{H}) \sum_{r=0}^{\infty} (1 - \frac{1}{2p(\mathcal{H})})^r S_r(\mathcal{G}, k) \leq 30736. \tag{104}$$

Consider now the H-graph $\mathcal{H}^{\mathcal{C}\ell}_{\mathcal{G}} = (\mathcal{V}, \mathcal{L}^{\mathcal{C}\ell}_{\mathcal{G}})$ of maximal cliques of $\mathcal{G}$. Notice that every maximal clique of $\mathcal{G}$ is composed of 4 nodes, which correspond to the vertices of a square in the grid. We have that $D_i(\mathcal{H}^{\mathcal{C}\ell}_{\mathcal{G}}) = 4$ for all $i \in \mathcal{V}$. Moreover, one can check that

$M_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}\ell}) \geq 4$ for all $i \in \mathcal{V}$ so that $p(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}\ell}) \geq 16$. The corresponding bound on the number of any player k's updates in any better response path is

$$2p(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}\ell}) \sum_{r=0}^{\infty} (1 - \frac{1}{2p(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}\ell})})^r S_r(\mathcal{G}, k) \geq 253984, \qquad (105)$$

for all augmented grid graphs of sufficiently large size. Comparing equations (104) and (105) we see that by exploiting the minimal separability property of the game we obtain a much tighter upper bound, which improves by more than a factor 8 the result of [14, Theorem 5.3] for large graphs.

In the previous Example 7.4.2 both the graphical and separable representation of the game yield a constant bound on the number of updates of any player in a better response path, which is independent on the number of players. Exploiting separability of the game results in a smaller constant bound. More in general, the bounds are increasing in the number of players and in these cases the order of such dependences is crucial in assessing the quality of the bounds. We next propose an example to show how exploiting separability may improve the bounds also in this setting.

To this aim, we consider a family of hyper-coordination games (see Definition 7.1.2) on hypergraphs obtained from the family of powers [24] of the n-cycle graph, as by the following definition.

Definition 7.4.3. *For $k > 0$, the k-th power of the n-cycle graph is a graph $\mathcal{G}$ with node set $\mathcal{V} = \{0, \dots, n-1\}$ and with link set*

$$\mathcal{E} = \{\{i, j\} | \min(|i-j|, n-|i-j|) \leq k\}.$$

Notice that for $k = 1$, $\mathcal{G}$ is a n-cycle, while for $k \geq n/2$ $\mathcal{G}$ is a complete graph. For any such $\mathcal{G}$ we consider an integer $0 < h \leq k$ and define the two following hypergraphs: the H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ whose hyperlinks are the $h+1$-cliques of $\mathcal{G}$, and the H-graphs of the maximal cliques $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l} = (\mathcal{V}, \mathcal{C}l(\mathcal{G}))$, whose hyperlinks are the $k+1$-cliques of $\mathcal{G}$.

In the following Example 7.4.4 we consider homogeneous hyper-coordination games, while in Example 7.4.5 we will investigate the role of this assumption on the results.

Example 7.4.4 (Homogeneous hyper-coordination game)**.** *Consider a homogeneous hyper-coordination game u as in Definition 7.1.2 on the H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ as defined in Definition 7.4.3, which is the hypergraph of the $h+1$-cliques of the k-power of an n-cycle $\mathcal{G}$. Notice that the hyper-coordination game on $\mathcal{H}$ is graphical on $\mathcal{G}$, which coincides with $\mathcal{G}^{\mathcal{F}^{\mathcal{H}}}$. Moreover the potential function ϕ is separable on $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$ and has minimal H-graph*

$\mathcal{H}_\phi = \mathcal{H}$. *We next compare the result obtained from Theorem 7.4.1 when considering the minimal H-graph* $\mathcal{H}$ *for the potential* ϕ *or the H-graph* $\mathcal{H}_\mathcal{G}^{\mathcal{C}l}$ *of the maximal-cliques of* $\mathcal{G}$*, to show that* $\mathcal{H}$ *provides a better bound.*

We start with the case when $h < k < n/2$. *Notice that we can associate each clique* $\mathcal{C}$ *of* $\mathcal{G}$ *to exactly one node* $i_\mathcal{C}$ *in* $\mathcal{V}$*, which is the unique node of* $\mathcal{G}$ *such that* $\mathcal{C} \subseteq \{i_\mathcal{C}, i_\mathcal{C} + 1, \dots, i_\mathcal{C} + k\}$*, where the addition is taken modulo* n. *Moreover, for each node* $i \in \mathcal{V}$*, there are* $\binom{k}{h}$ $h+1$*-cliques and only one* $k+1$*-clique of* $\mathcal{G}$ *associated to* i. *It then follows that the number of* $h+1$*-cliques of* $\mathcal{G}$ *is* $|\mathcal{L}| = n\binom{k}{h}$*, while the number of* $k+1$*-cliques is* $|\mathcal{L}_\mathcal{G}^{\mathcal{C}l}| = |\mathcal{C}l(\mathcal{G})| = n$. *We can use this information to compute the H-degrees* $D_i(\mathcal{H})$ *and* $D_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l})$*, which do not depend on the player* i. *We have that for all* $i \in \mathcal{V}$

$$
\begin{aligned}
D_i(\mathcal{H}) &= \frac{|\mathcal{L}|(h+1)}{n} = (h+1)\binom{k}{h} \\
D_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) &= \frac{|\mathcal{L}_\mathcal{G}^{\mathcal{C}l}|(k+1)}{n} = k+1.
\end{aligned}
$$

To compute the maximum of local potential functions $M_i(\mathcal{H})$ *and* $M_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l})$ *(that are independent on* i*), we first observe that from (82) and the homogeneity assumption*

$$
M_i(\mathcal{H}) = \max_{a \in \mathcal{A}} w(a). \tag{106}
$$

We then construct a decomposition of ϕ *on* $\mathcal{H}_\mathcal{G}^{\mathcal{C}l}$*: for each* $k+1$*-clique* $\mathcal{J} \in \mathcal{L}_\mathcal{G}^{\mathcal{C}l}$*, we define the local potential* $\phi_\mathcal{J}$ *as*

$$
\phi_\mathcal{J}(x) = \sum_{\mathcal{K} \in \mathcal{L} : i_\mathcal{J} = i_\mathcal{K}} \bar{w}(x, \mathcal{K}),
$$

which is the sum of local potential in the minimal decomposition (82) corresponding to $h+1$*-cliques associated to the same player as the* $k+1$*-clique* $\mathcal{J}$. *This shows that for any player* i*, denoting with* $\mathcal{J}$ *the* $k+1$ *clique such that* $i = i_\mathcal{J}$*, we have that*

$$
M_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) = \sum_{\mathcal{K} \in \mathcal{L} : i_\mathcal{K} = i} M_i(\mathcal{H}) = \binom{k}{h} M_i(\mathcal{H}),
$$

where the first equality is realised for the configuration $x^* = \mathbf{1} \arg\max_{a \in \mathcal{A}} w(a)$. *In conclusion,*

$$
p(\mathcal{H}) = (h+1)\binom{k}{h} \max_{a \in \mathcal{A}_i} w(a) < (k+1)\binom{k}{h} \max_{a \in \mathcal{A}_i} w(a) = p(\mathcal{H}_\mathcal{G}^{\mathcal{C}l})
$$

as $h < k$*, showing that the bound of Theorem 7.4.1 is lower for* $\mathcal{H}$ *than for* $\mathcal{H}_\mathcal{G}^{\mathcal{C}l}$.

We now complete the analysis by dealing with the remaining cases when $k \geq n/2$*, i.e., when* $\mathcal{H}$ *is the hypergraph of the* $h+1$*-cliques of the complete graph* $\mathcal{G}$ *with* n *nodes. We* *immediately have that* $\mathcal{L} = \{\mathcal{K} \subset \mathcal{V} : |\mathcal{K}| = h+1\}$ *with* $|\mathcal{L}| = \binom{n}{h+1}$ *while* $|\mathcal{L}_{\mathcal{G}}^{\mathcal{C}l}| = 1$ *since* $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$ *possess a unique hyperlink containing all nodes* $\mathcal{V}$*. We can easily compute the* H*-degrees* $D_i(\mathcal{H})$ *and* $D_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l})$*, which do not depend on the player* i*:*

$$D_i(\mathcal{H}) = \binom{n-1}{h}$$
$$D_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}) = 1, \qquad \forall i \in \mathcal{V}.$$

To compute the largest maximum values of the local potentials $M_i(\mathcal{H})$ *and* $M_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l})$ *we compare the minimal decomposition* (82) *of* ϕ *given by* $\mathcal{H}$ *to the trivial decomposition of* ϕ *over* $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$*, thus obtaining that:*

$$M_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}) = \max_{x \in \mathcal{X}} \phi(x) = \max_{x \in \mathcal{X}} \sum_{\mathcal{K} \in \mathcal{L}} \bar{w}(x, \mathcal{K}) = \binom{n}{h+1} M_i(\mathcal{H}).$$

where the last equality is realised for the configuration $x^* = \mathbf{1} \arg\max_{a \in \mathcal{A}} w(a)$*. It then follows that*

$$p_i(\mathcal{H}) = \binom{n-1}{h} M_i(\mathcal{H}), \qquad p_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}) = \binom{n}{h+1} M_i(\mathcal{H}),$$

which implies that

$$\frac{p_i(\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l})}{p_i(\mathcal{H})} = \frac{\binom{n}{h+1}}{\binom{n-1}{h}} = \frac{n}{h+1} > 1$$

since $h+1 < n$*. As a consequence, the bound of Theorem* 7.4.1 *with* $\mathcal{H} = \mathcal{H}_\phi$ *is lower than the one obtained with* $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$*.*

Example 7.4.4 shows that considering the minimal H-graph $\mathcal{H}_\phi = \mathcal{H}$ in Theorem 7.4.1 improves the result of [14, Theorem 5.3] for all homogeneous coordination games on hypergraphs originated from powers of a cycle graph. In the following example we show that the homogeneity assumption is crucial for such result. More in general, this shows how a finer separability of the potential function ϕ than the one deriving from the graphical representation of games cannot always be exploited to obtain a better bound.

Example 7.4.5. *Consider the* H*-graph* $\mathcal{H} = (\mathcal{V}, \mathcal{L})$*, where* $\mathcal{V} = \{1, 2, 3\}$ *and* $\mathcal{L} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ *with* $\mathcal{K}_j = \mathcal{V} \setminus \{j\}$*. Moreover, consider the coordination game on* $\mathcal{H}$ *with action set* $\mathcal{A} \equiv \mathcal{V} = \{1, 2, 3\}$ *and weight function*

$$w(a, \mathcal{K}_j) = \begin{cases} 1 & \text{if } a = j \\ 0 & \text{otherwise} \end{cases}, \qquad a \in \mathcal{A}, \mathcal{K}_j \in \mathcal{L}.$$

Notice that the game can equivalently be described as a pairwise graphical game on a 3-clique (or 3-cycle) $\mathcal{G}$, where on each link two players aim at coordinating on one of the three actions (namely, the one corresponding to the third player). Besides the minimal H-graph $\mathcal{H}_\phi \equiv \mathcal{H}$ of the potential function, we can also consider the cliques H-graph $\mathcal{H}_\mathcal{G}^{\mathcal{C}l} = (\mathcal{V}, \mathcal{L}_\mathcal{G}^{\mathcal{C}l})$, which only possess one hyperlink:

$$\mathcal{L}_\mathcal{G}^{\mathcal{C}l} = \{\{1,2,3\}\}.$$

As a consequence, for all $i \in \mathcal{V}$

$$\begin{aligned} D_i(\mathcal{H}) &= 2 & D_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) &= 1 \\ M_i(\mathcal{H}) &= 1 & M_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) &= 1 \\ p(\mathcal{H}) &= 2 & p(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) &= 1. \end{aligned}$$

Notice that $M_i(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) = 1$ since in any configuration x at most one couple of distinct players (i,j) may coordinate on an action $k \in \mathcal{V} \setminus \{i,j\}$. Then, for such game we have $p(\mathcal{H}_\mathcal{G}^{\mathcal{C}l}) < p(\mathcal{H}_\phi)$, so that the bound of Theorem 7.4.1 with $\mathcal{H} = \mathcal{H}_\mathcal{G}^{\mathcal{C}l}$ is lower than the one obtained with $\mathcal{H} = \mathcal{H}_\phi$.

7.4.2 *Appendix: proof of Theorem 7.4.1*

Proof of 7.4.1. Let $\lambda = 1 - \frac{1}{2p}$. Fix a node k in $\mathcal{V}$, and define the potential

$$\Theta(x) = \sum_{\mathcal{J} \in \mathcal{L}} \lambda^{\Delta(k,\mathcal{J})} \phi_\mathcal{J}(x_\mathcal{J}),$$

where $\Delta(k,\mathcal{J}) = \min_{h \in \mathcal{J}} \Delta(k,h)$. Θ is an ordinal potential [69] for the game, i.e., it is non-decreasing along any better-response path. Indeed, consider any $x \in \mathcal{X}$ and $y \sim_i x \in \mathcal{X}$ for some $i \in \mathcal{V}$ such that $x \neq y$, and assume that $\phi(y) > \phi(x)$. Since x and y only differ for the action of player i, then

$$\phi(y) - \phi(x) = \sum_{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}} [\phi_\mathcal{J}(y) - \phi_\mathcal{J}(x)],$$

which implies

$$\Theta(y) - \Theta(x) = \sum_{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}} \lambda^{\Delta(k,\mathcal{J})} [\phi_\mathcal{J}(y) - \phi_\mathcal{J}(x)].$$

Notice that for any hyperlink $\mathcal{J}$ that includes i it holds that either $\Delta(k,\mathcal{J}) = \Delta(k,i)$ or $\Delta(k,\mathcal{J}) = \Delta(k,i) - 1$. We denote

$$\mathcal{J}_1 = \{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}, \Delta(k,\mathcal{J}) = \Delta(k,i)\}$$

and

$$\mathcal{J}_2 = \{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}, \Delta(k, \mathcal{J}) = \Delta(k, i) - 1\}$$

so that we can write

$$\begin{aligned}
\frac{\Theta(y) - \Theta(x)}{\lambda^{\Delta(k,i)-1}} &= \lambda \sum_{\mathcal{J} \in \mathcal{J}_1} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] + \sum_{\mathcal{J} \in \mathcal{J}_2} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] \\
&= \sum_{\mathcal{J} \in \mathcal{L} : i \in \mathcal{J}} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] - (1 - \lambda) \sum_{\mathcal{J} \in \mathcal{J}_1} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] \\
&= \phi(y) - \phi(x) - (1 - \lambda) \sum_{\mathcal{J} \in \mathcal{J}_1} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] \\
&= \phi(y) - \phi(x) - \frac{1}{2p} \sum_{\mathcal{J} \in \mathcal{J}_1} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] \\
&\geq 1 - \frac{1}{2p} \sum_{\mathcal{J} \in \mathcal{J}_1} [\phi_{\mathcal{J}}(y) - \phi_{\mathcal{J}}(x)] \geq 0,
\end{aligned}$$

since the last sum has at most D_i terms, each of which is strictly less than $2M_i$. By the definition of Θ we have that for $x \in \mathcal{X}$

$$\begin{aligned}
|\Theta(x)| &= \left| \sum_{\mathcal{J} \in \mathcal{L}} \lambda^{\Delta(k,\mathcal{J})} \phi_{\mathcal{J}}(x_{\mathcal{J}}) \right| \\
&= \left| \sum_{r=0}^{\infty} \sum_{\substack{\mathcal{J} \in \mathcal{L} \\ \Delta(k,\mathcal{J}) = r}} \lambda^r \phi_{\mathcal{J}}(x_{\mathcal{J}}) \right| \\
&\leq \sum_{r=0}^{\infty} \lambda^r \sum_{\substack{h \in \mathcal{V} \\ \Delta(h,k) = r}} \sum_{\substack{\mathcal{J} \in \mathcal{L} \\ h \in \mathcal{J}}} |\phi_{\mathcal{J}}(x_{\mathcal{J}})| \\
&\leq \sum_{r=0}^{\infty} \lambda^r \sum_{\substack{h \in \mathcal{V} \\ \Delta(h,k) = r}} \underbrace{D_h M_h}_{\leq p} \\
&\leq p \sum_{r=0}^{\infty} \lambda^r S_r(\mathcal{G}, k).
\end{aligned}$$

Every better response move of player k causes an increment by at least 1 (because the potential is integer) of

$$\sum_{\mathcal{J} \in \mathcal{L} : \Delta(k,\mathcal{J}) = 0} \phi_{\mathcal{J}} = \sum_{\mathcal{J} \in \mathcal{L} : \Delta(k,\mathcal{J}) = 0} \lambda^{\Delta(k,\mathcal{J})} \phi_{\mathcal{J}}$$

and causes no change in

$$\sum_{\mathcal{J}\in\mathcal{L}:\Delta(k,\mathcal{J})\geq 1} \lambda^{\Delta(k,\mathcal{J})}\phi_{\mathcal{J}},$$

which implies an increment by at least 1 of Θ. Finally, since Θ is non-decreasing along a better response path, the total number of times that player k updates their strategy is at most

$$2p\sum_{r=0}^{\infty}\left(1-\frac{1}{2p}\right)^{r} S_r(\mathcal{G},k).$$

□

8 GEOMETRIC CHARACTERIZATION OF SEPARABILITY

Thanks to the results of Chapter 5, we can already characterize separability of games in terms of projections. More precisely, a game $u \in \mathcal{U}$ is $\mathcal{F}$-separable if and only if it coincides with its projection onto the space of $\mathcal{F}$-separable games, namely

$$\Pi_{\mathcal{F}} u = u. \tag{107}$$

This equation expresses separability of a game with a system of linear constraints, one for each player $i \in \mathcal{V}$ and configuration $x \in \mathcal{X}$ of the game

$$u_i(x) = u_{\mathcal{F},i}(x). \tag{108}$$

Despite being simple, this formulation is not very expressive. Indeed, each of the equations (108) involves all the values of utility function u_i and, more importantly, it tangles the contributions of all hyperlinks of $\mathcal{F}$. This becomes apparent if we expand (108) with the explicit expression of the projector $\Pi_{\mathcal{F}}$ obtained in (64)

$$u_i(x) = (u_{\mathcal{F}})_i(x) = \sum_{\substack{S \subset \mathcal{L}_i \\ S \neq \emptyset}} (-1)^{|S|+1} \frac{1}{|\mathcal{X}_{\mathcal{V} \setminus \cap S}|} \sum_{y \in \mathcal{X}_{\mathcal{V} \setminus \cap S}} u_i(x_{\cap S}, y).$$

For this reason, this characterization falls short when one tries to construct the minimal FDH-graph of a given game. Indeed, by using (108) alone, it appears that one would need to search exhaustively over all FDH-graphs $\mathcal{F}$ until the equation is satisfied, which is not a meaningful approach. A related problem is that of checking whether a provided decomposition of a game's utilities is minimal. Clearly, the approach based on projections is not optimal, as it requires to check that equation (107) is satisfied for the given FDH-graph $\mathcal{F}$ while it is not satisfied by any finer FDH-graph $\tilde{\mathcal{F}} \prec \mathcal{F}$, which usually accounts for a large number of checks.

To address these points, this chapter is dedicated to deriving a different characterization of separability of games. The objective is to express separability of a game u in terms of explainable constraints, which isolate the influence of each hyperlink.

The proposed characterization is of geometric nature, as it leverages the topology of the configuration space $\mathcal{X}$ operating on cubic subgraphs of the configuration graph $\mathcal{G}_{\text{conf}}$ (see Section 2.4.2).

Thanks to this characterization we obtain checkable conditions to identify the minimal FDH-graph of a game. This results in an algorithm whose efficiency improves on exhaustive search and, more importantly, provides insights into the meaning of separability. Notice that, once the minimal FDH-graph $\mathcal{F}$ of a game u is known, the $\mathcal{F}$-separable representation of the game (i.e., the explicit decomposition of its utility functions) can be obtained exploiting projections by computing $u = \Pi_{\mathcal{F}} u$.

8.1 GEOMETRIC CHARACTERIZATION OF HIGH ORDER INTERACTIONS

In this section we derive a characterization of high-order interactions in games. The results of this section will be the basis to develop algorithmic procedures to identify the hypergraph structure of games, which will be accomplished in the next sections.

Let $\mathcal{K} = \{j_1, j_2, \dots, j_k\} \subset \mathcal{V}$ be a set of players, with $k = |\mathcal{K}|$. For any choice of a couple of actions for each player in $\mathcal{K}$, $\{\{\alpha_{j_1}, \beta_{j_1}\}, \{\alpha_{j_2}, \beta_{j_2}\}, \dots, \{\alpha_{j_k}, \beta_{j_k}\}\}$, and for each configuration of the remaining players $x_{-\mathcal{K}} \in \mathcal{X}_{-\mathcal{K}}$, we can construct the cube of dimension k in the configuration space $\mathcal{X}$ with vertices y corresponding to the carthesian product $\{\alpha_{j_1}, \beta_{j_1}\} \times \{\alpha_{j_2}, \beta_{j_2}\} \times \dots \times \{\alpha_{j_k}, \beta_{j_k}\}$, i.e., the vertices y coincide with $x_{-\mathcal{K}}$ for actions of players in $\mathcal{V} \setminus \mathcal{K}$ and to all possible combinations of one action per player $\ell \in \mathcal{K}$, chosen among the selected couples $\{\alpha_\ell, \beta_\ell\}$. So, formally,

Definition 8.1.1. *A cube Q associated to $\mathcal{K} = \{j_1, j_2, \dots, j_k\} \subset \mathcal{V}$ (also referred to as a $\mathcal{K}$-cube) is a set of configurations:*

$$\{y \in \mathcal{X} \,|\, y_{-\mathcal{K}} = x_{-\mathcal{K}}, y_\ell \in \{\alpha_\ell, \beta_\ell\}, \forall \ell \in \mathcal{K}\}\,.$$

where $\{\{\alpha_{j_1}, \beta_{j_1}\}, \{\alpha_{j_2}, \beta_{j_2}\}, \dots, \{\alpha_{j_k}, \beta_{j_k}\}\}$ is a choice of a couple of actions for each player in $\mathcal{K}$ and $x_{-\mathcal{K}}$ is a fixed configuration of players in $\mathcal{V} \setminus \mathcal{K}$.

Given a $\mathcal{K}$-cube Q, its vertices can be labelled with boolean vectors $b \in \{0,1\}^{\mathcal{K}}$. This allows to compactly represent vertices of Q as $\{x_Q^b\}_{b \in \{0,1\}^K}$ where for each player $j \in \mathcal{K}$

$$(x_Q^b)_j = \begin{cases} \alpha_j & \text{if } b_j = 0 \\ \beta_j & \text{if } b_j = 1 \end{cases} \qquad (109)$$

while $(x_Q^b)_{-\mathcal{K}} = x_{-\mathcal{K}}$

Example 8.1.2. *Consider a game with 4 players $\mathcal{V} = \{1,2,3,4\}$ and binary actions $\mathcal{A} = \{0,1\}$. Fix a subset of players $\mathcal{K} = \{1,2\}$. Figure 34 represents the $\mathcal{K}$-cube associated with the configuration $x_{-\mathcal{K}}$ such that $(x_{-\mathcal{K}})_3 = 1$ and $(x_{-\mathcal{K}})_4 = 0$.*

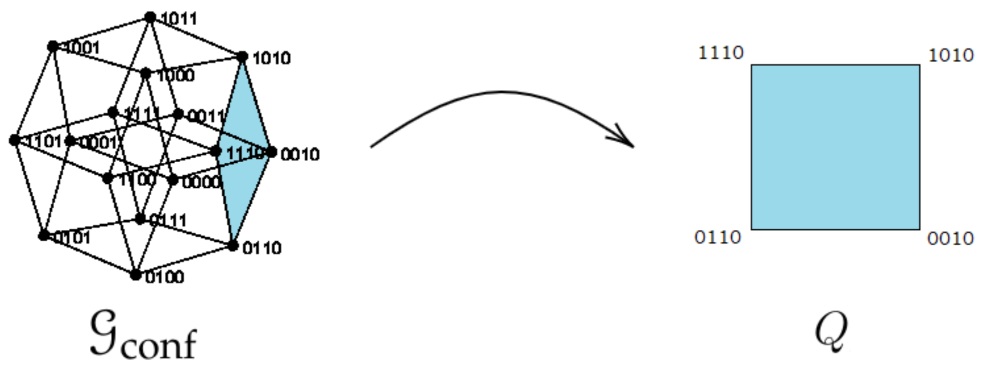


Figure 34: A cube Q for a 4-players game with binary actions 0 and 1, as discussed in Example 8.1.2. The configuration graph $\mathcal{G}_{\text{conf}}$ of the game is represented on the left, with the cube Q highlighted in blue.

Based on this definition, the following lemma characterize functions $f \in \mathbb{R}^{\mathcal{X}}$ that *do not* jointly depend on a set of variables $\mathcal{K} \subset \mathcal{V}$. To this aim, for a set $\mathcal{K} \subset \mathcal{V}$, we define the H-graph $\mathcal{H}^{\mathcal{K}} = (\mathcal{V}, \mathcal{L})$ by

$$\mathcal{L} = \{\mathcal{V} \setminus \{h\} : h \in \mathcal{K}\}.$$

Lemma 8.1.3. *Let $f \in \mathbb{R}^{\mathcal{X}}$ and $\mathcal{K} \subset \mathcal{V}$. f is $\mathcal{H}^{\mathcal{K}}$-separable if and only if for all $\mathcal{K}$-cubes Q:*

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} f(x_Q^b) = 0. \tag{110}$$

Proof. We start by proving necessity. By assumption f is $\mathcal{H}^{\mathcal{K}}$-separable so that we may write

$$f(x) = \sum_{i \in \mathcal{K}} f_{\mathcal{V} \setminus \{i\}}(x_{\mathcal{V} \setminus \{i\}}).$$

By linearity it is sufficient to prove that for all $i \in \mathcal{K}$

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} f_{\mathcal{V} \setminus \{i\}}((x_Q^b)_{\mathcal{V} \setminus \{i\}}) = 0.$$

This can be proven by observing that $(x_Q^b)_{\mathcal{V}\setminus\{i\}}$ does not depend on b_i:

$$\begin{aligned}
&\sum_{b\in\{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} f_{\mathcal{V}\setminus\{i\}}((x_Q^b)_{\mathcal{V}\setminus\{i\}}) \\
&= \sum_{\substack{b\in\{0,1\}^{\mathcal{K}}\\ b_i=0}} (-1)^{\mathbb{1}'b} f_{\mathcal{V}\setminus\{i\}}((x_Q^b)_{\mathcal{V}\setminus\{i\}}) + \sum_{\substack{b\in\{0,1\}^{\mathcal{K}}\\ b_i=1}} (-1)^{\mathbb{1}'b} f_{\mathcal{V}\setminus\{i\}}((x_Q^b)_{\mathcal{V}\setminus\{i\}}) \\
&= \sum_{\substack{b\in\{0,1\}^{\mathcal{K}}\\ b_i=0}} (-1)^{\mathbb{1}'b} f_{\mathcal{V}\setminus\{i\}}((x_Q^b)_{\mathcal{V}\setminus\{i\}}) - \sum_{\substack{b\in\{0,1\}^{\mathcal{K}}\\ b_i=0}} (-1)^{\mathbb{1}'b} f_{\mathcal{V}\setminus\{i\}}((x_Q^b)_{\mathcal{V}\setminus\{i\}}) = 0.
\end{aligned}$$

We now prove sufficiency. To do this we show that f coincides with its $\mathcal{H}^{\mathcal{K}}$-separable projection $f_{\mathcal{H}^{\mathcal{K}}}$. By applying formulas (66) and (65) we have that

$$\begin{aligned}
f_{\mathcal{H}^{\mathcal{K}}}(x) &= \sum_{\substack{\mathcal{J}\subset\mathcal{K}\\ \mathcal{J}\neq\emptyset}} (-1)^{|\mathcal{J}|+1} \Pi_{\mathcal{V}\setminus\mathcal{J}} f(x) \\
&= \sum_{\substack{\mathcal{J}\subset\mathcal{K}\\ \mathcal{J}\neq\emptyset}} (-1)^{|\mathcal{J}|+1} \frac{1}{|\mathcal{X}_{\mathcal{J}}|} \sum_{y_{\mathcal{J}}\in\mathcal{X}_{\mathcal{J}}} f(x_{\mathcal{V}\setminus\mathcal{J}}, y_{\mathcal{J}}) \\
&= \sum_{\substack{y_{\mathcal{K}}\in\mathcal{X}_{\mathcal{K}}\\ y_{\mathcal{K}}\neq x_{\mathcal{K}}}} f(x_{\mathcal{V}\setminus\mathcal{K}}, y_{\mathcal{K}}) (-1)^{|\mathcal{J}_{y_{\mathcal{K}}}|+1} \frac{1}{|\mathcal{X}_{\mathcal{J}_{y_{\mathcal{K}}}}|}
\end{aligned}$$

where $\mathcal{J}_{y_{\mathcal{K}}} = \{i \in \mathcal{K} : (y_{\mathcal{K}})_i \neq x_i\}$. We then have that

$$\begin{aligned}
f_{\mathcal{H}^{\mathcal{K}}}(x) &= \sum_{\substack{\mathcal{J}\subset\mathcal{K}\\ \mathcal{J}\neq\emptyset}} (-1)^{|\mathcal{J}|+1} \frac{1}{|\mathcal{X}_{\mathcal{J}}|} \sum_{\substack{y_{\mathcal{K}}\in\mathcal{X}_{\mathcal{K}}\\ \mathcal{J}_{y_{\mathcal{K}}}=\mathcal{J}}} f(x_{\mathcal{V}\setminus\mathcal{K}}, y_{\mathcal{K}}) \\
&= f(x) + \sum_{\mathcal{J}\subset\mathcal{K}} (-1)^{|\mathcal{J}|+1} \frac{1}{|\mathcal{X}_{\mathcal{J}}|} \sum_{\substack{y_{\mathcal{K}}\in\mathcal{X}_{\mathcal{K}}\\ \mathcal{J}_{y_{\mathcal{K}}}=\mathcal{J}}} f(x_{\mathcal{V}\setminus\mathcal{K}}, y_{\mathcal{K}}) \\
&= f(x) - \frac{1}{|\mathcal{X}_{\mathcal{K}}|} \sum_{\mathcal{J}\subset\mathcal{K}} \sum_{\substack{y_{\mathcal{K}}\in\mathcal{X}_{\mathcal{K}}\\ \mathcal{J}_{y_{\mathcal{K}}}=\mathcal{J}}} (-1)^{|\mathcal{J}|} |\mathcal{X}_{\mathcal{K}\setminus\mathcal{J}}| f(x_{\mathcal{V}\setminus\mathcal{K}}, y_{\mathcal{K}}) \\
&= f(x) - \frac{1}{|\mathcal{X}_{\mathcal{K}}|} \sum_{\substack{Q\ \mathcal{K}\text{-cube}\\ x_Q^0=x}} \sum_{b\in\{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} f(x_Q^b) \\
&= f(x),
\end{aligned}$$

which concludes the proof.

□

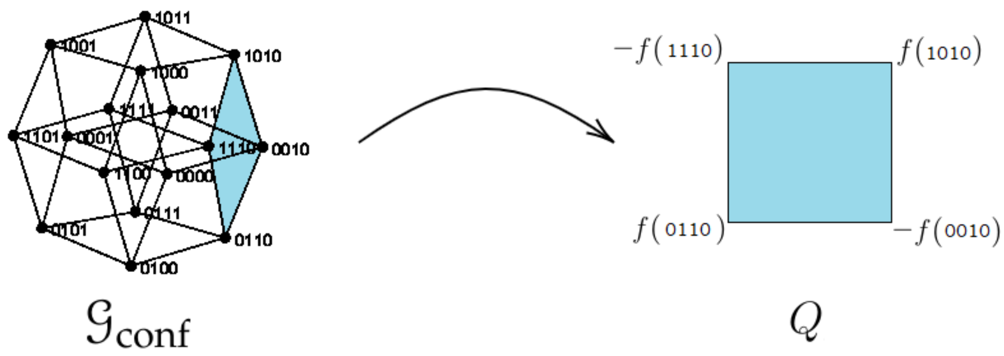


Figure 35: One of the checks to be performed to assess separability of a function f in the setting of Example 8.1.4, as by Lemma 8.1.3.

Example 8.1.4. *Consider again a game with* 4 *players* $\mathcal{V} = \{1, 2, 3, 4\}$ *and binary actions* $\mathcal{A} = \{0, 1\}$*. Figure 35 describes the check prescribed by Lemma 8.1.3 for a function* f *and* $\mathcal{K}$*-cube* Q*, where* $\mathcal{K} = \{1, 2\}$*. Values of* f *at the vertices of* Q *must be linearly combined, with the appropriate signs determined by vertex labels.*

Based on this, we are able to characterize the separability of a function $f \in \mathbb{R}^{\mathcal{X}}$.

Lemma 8.1.5. *Let* $f \in \mathbb{R}^{\mathcal{X}}$ *and* $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ *be an H-graph.* f *is* $\mathcal{H}$*-separable if and only if for all* $\mathcal{K} \subset \mathcal{V}$ *such that* $\mathcal{K} \not\subset \mathcal{J}, \forall \mathcal{J} \in \mathcal{L}$ *and for all* $\mathcal{K}$*-cubes* Q *it holds that*

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} f(x_Q^b) = 0. \tag{111}$$

Proof. We first prove the only if implication. By applying Lemma 8.1.3 we obtain that f is $\mathcal{H}^{\mathcal{K}}$-separable for all $\mathcal{K} \subset \mathcal{V}$ such that $\forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}$. As a consequence, f is separable with respect to the intersection

$$\mathcal{H}' = \sqcap_{\substack{\mathcal{K} \subset \mathcal{V}: \\ \forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}}} \mathcal{H}^{\mathcal{K}} := (\mathcal{V}, \mathcal{L}'). \tag{112}$$

We then show that $\mathcal{H} \succeq \mathcal{H}'$. Let $\mathcal{J}' \in \mathcal{L}'$. For each $\mathcal{K} \subset \mathcal{V}$ such that $\forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}$, there exists $h^{\mathcal{K}} \in \mathcal{K}$ such that

$$\begin{aligned} \mathcal{J}' &= \bigcap_{\substack{\mathcal{K} \subset \mathcal{V}: \\ \forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}}} \mathcal{V} \setminus \{h^{\mathcal{K}}\} \\ &= \mathcal{V} \setminus \{h^{\mathcal{K}} : \mathcal{K} \subset \mathcal{V} \text{ s.t. } \forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}\}. \end{aligned}$$

If for all $\mathcal{J} \in \mathcal{L}$, $\mathcal{J}' \not\subset \mathcal{J}$, then there exists $h^{\mathcal{J}'} \in \mathcal{J}'$ belonging to the set

$$\{h^{\mathcal{K}} : \mathcal{K} \subset \mathcal{V} \text{ s.t. } \forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}\}.$$

But this implies that $h^{\mathcal{J}'} \notin \mathcal{J}'$, which is a contradiction. Then $\mathcal{H}' \preceq \mathcal{H}$ and since f is $\mathcal{H}'$-separable it follows that f is $\mathcal{H}$-separable.

We now prove the reverse implication. If f is $\mathcal{H}$-separable, then it is also $\mathcal{H}^{\mathcal{K}}$-separable for all $\mathcal{K} \subset \mathcal{V}$ such that $\forall \mathcal{J} \in \mathcal{L}, \mathcal{K} \not\subset \mathcal{J}$. From this, by applying Lemma 8.1.3, we obtain (112). □

Finally, building on Lemma 8.1.5 we obtain the following characterization of separable games.

Theorem 8.1.6. *Let $u \in \mathcal{U}$ and $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ be an FDH-graph. u is $\mathcal{F}$-separable if and only if for all $i \in \mathcal{V}$, and for all $\mathcal{K} \subset \mathcal{V}$ such that $i \in \mathcal{K}$ and $\forall (i, \mathcal{J}) \in \mathcal{D}$, $\mathcal{K} \not\subset \{i\} \cup \mathcal{J}$, it holds that for all $\mathcal{K}$-cubes Q*

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} u_i(x_Q^b) = 0. \tag{113}$$

Proof. The statement follows from Lemma 8.1.5 and from the fact that u is $\mathcal{F}$-separable if and only if for each $i \in \mathcal{V}$ the utility function u_i is $\mathcal{H}_i$-separable with respect to the local H-graph $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ introduced in (40) where

$$\mathcal{L}_i = \{\mathcal{V} \setminus \{i\}\} \bigcup \{\{i\} \cup \mathcal{J} : (i, \mathcal{J}) \in \mathcal{D}\}.$$

□

Separability of games can be equivalently characterized in terms of flows (see Section 2.4.2), as by the following result.

Corollary 8.1.7. *Let $u \in \mathcal{U}$, $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ be an FDH-graph and denote with $F = Du \in F\ell$ the flow associated to u. u is $\mathcal{F}$-separable if and only if for all $i \in \mathcal{V}$ and for all $\mathcal{K} \subset \mathcal{V} \setminus \{i\}$ such that $\mathcal{K} \not\subset \mathcal{J}$, $\forall (i, \mathcal{J}) \in \mathcal{D}$, it holds that for all $\mathcal{K}$-cubes Q and actions $y_i \in \mathcal{A}_i$*

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} F(x_Q^b, ((x_Q^b)_{-i}, y_i) = 0 \tag{114}$$

Proof. This follows directly from Theorem 8.1.6 and from the definition of the operator D given in 2.4.10. □

Theorem 8.1.6 characterizes separability. By also accounting for minimality we can obtain the following check to tell whether an FDH-graph is minimal for a given game.

Corollary 8.1.8. *Let $u \in \mathcal{U}$ and $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ be a simple FDH-graph such that u is $\mathcal{F}$-separable. $\mathcal{F}$ is the minimal FDH-graph of u if and only if for all $\forall (i, \mathcal{J}) \in \mathcal{D}$, by letting $\mathcal{K} = \{i\} \cup \mathcal{J}$, there exists a $\mathcal{K}$-cube Q such that*

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} u_i(x_Q^b) \neq 0. \tag{115}$$

Proof. Assume that for all $(i, \mathcal{J}) \in \mathcal{D}$ equation (115) is satisfied by some cube associated to $\mathcal{K} = \{i\} \cup \mathcal{J}$. Consider an FDH-graph $\mathcal{F}' = (\mathcal{V}, \mathcal{D}')$ such that $\mathcal{F}' \prec \mathcal{F}$. We can show that u is not $\mathcal{F}'$-separable as the characterization of Theorem 8.1.6 fails for u and $\mathcal{F}'$ when considering equation (113) for $\mathcal{K} = \{i\} \cup \mathcal{J}$ and $(i, \mathcal{J}) \in \mathcal{D} \setminus \mathcal{D}'$. Conversely, assume that $\mathcal{F}$ is the minimal FDH-graph for u. We show that equation (115) is satisfied for all $\mathcal{K} = \{i\} \cup \mathcal{J}$ and $(i, \mathcal{J}) \in \mathcal{D}$. Suppose by contradiction that $\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} u_i(x_Q^b) = 0$ for all cubes Q associated to some $\mathcal{K} = \{i\} \cup \mathcal{J}$, with $(i, \mathcal{J}) \in \mathcal{D}$. Then we can construct an FDH-graph $\mathcal{F}' = (\mathcal{V}, \mathcal{D}')$ such that $\mathcal{F}' \prec \mathcal{F}$ and u is $\mathcal{F}'$-separable. More precisely, by setting $\mathcal{D}' = (\mathcal{D} \setminus \{(i, \mathcal{J})\}) \cup \{(i, \mathcal{J} \setminus \{k\}) : k \in \mathcal{J}\}$, the FDH-graph $\mathcal{F}'$ satisfies the characterization of Theorem 8.1.6, contradicting the minimality of $\mathcal{F}$. □

8.2 MINIMAL SEPARABILITY ALGORITHMS

Let u be a game with $n = |\mathcal{V}|$ players. Given a subset of players $\mathcal{K} \subset \mathcal{V}$ and a player $i \in \mathcal{V}$ we are interested in determining whether player i jointly depends on the members of $\mathcal{K}$. By calling $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ the minimal FDH-graph of u, this is equivalent to understanding whether $\mathcal{F}$ contains any hyperlink $(i, \mathcal{J})$ such that $\mathcal{K} \subset \{i\} \cup \mathcal{J}$.

The following holds as a direct consequence of Corollary 8.1.8.

Proposition 8.2.1. *Consider a player $i \in \mathcal{V}$ and a set of players $\mathcal{K} \subset \mathcal{V}$ such that $i \in \mathcal{K}$. The minimal FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ of u possesses an hyperlink $(i, \mathcal{J}) \in \mathcal{D}$ such that $\mathcal{K} \subset \mathcal{J} \cup \{i\}$ if and only if the following is satisfied*

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} u_i(x_Q^b) \neq 0 \tag{116}$$

for some cube Q associated to $\mathcal{K}$.

Note that equation (116) involves a linear combination of the values of utility u_i at the vertices of cubes associated with players $\mathcal{K}$, with coefficients $+1$ or -1 which alternate along edges of the cube.

Proposition 8.2.1 suggests a procedure to construct the minimal hypergraph $\mathcal{F}$ for a game u. An algorithm for solving this task is described below.

Algorithmus 1 : Compute the minimal FDH-graph of a game.

Result : minimal FDH-graph $\mathcal{F}$ of u
Initialization: set $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ with $\mathcal{D} = \emptyset$;
foreach $i \in \mathcal{V}$ **do**
  set $\mathcal{L} = \emptyset$;
  foreach $\mathcal{K} \subset \mathcal{V}$, *s.t.* $i \in \mathcal{K}$ **do**
    foreach $\mathcal{K}$*-cube* Q **do**
      if $\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} u_i(x_Q^b) \neq 0$ **then**
        add $\mathcal{K}$ to $\mathcal{L}$;
      end
    end
  end
  foreach *maximal set* $\mathcal{J} \in \mathcal{L}$ **do**
    add the hyperlink $(i, \mathcal{J} \setminus \{i\})$ to $\mathcal{D}$;
  end
end

The algorithm constructs, for each player $i \in \mathcal{V}$, the hyperlinks $(i, \mathcal{J})$ of $\mathcal{F}$ with tail node i. For doing this, for each $\mathcal{K} \subset \mathcal{V}$ such that $i \in \mathcal{K}$ it checks whether u_i jointly depends on the actions of players in $\mathcal{K} \setminus \{i\}$. If so, the set $\mathcal{K}$ is kept and collected in $\mathcal{L}$, otherwise it is discarded. The check on $\mathcal{K}$ is performed by iterating over all cubes associated with $\mathcal{K}$ in the configuration space and by testing Equation (116) for each cube. When all sets $\mathcal{K} \subset \mathcal{V}$ containing i have been checked, the head sets $\mathcal{J}$ of hyperlinks starting at i are simply obtained from the maximal sets of $\mathcal{L}$. This task is repeated for each player.

For the case of potential games, the symmetry of their structure can be exploited to refine the general Algorithm 1 for constructing the minimal FDH-graph of the game. Indeed, Theorem 7.1.1 showed that the minimal FDH-graph with respect to which a potential game is separable has a symmetry property and is completely determined by the H-graph describing the minimal separability of the potential function itself. So, for a potential game u, the problem of computing the minimal FDH-graph $\mathcal{F}$ is reduced to computing the minimal hypergraph $\mathcal{H}$ of its potential function. The following algorithm provides a procedure for completing this task.

Algorithmus 2 : Compute the minimal FDH-graph of a potential game.

Result : Minimal hypergraph $\mathcal{F}$ of u
Initialization: set $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ with $\mathcal{D} = \emptyset$;
set $\phi \in \mathbb{R}^{\mathcal{X}}$ to $\mathbf{0}$;
enumerate configurations in $\mathcal{X}$ in lexicographic order ;
set $\phi = 0$ on the first configuration x^0 ;
foreach $k \in \{0, \ldots, |\mathcal{X}| - 1\}$ **do**
  construct x^{k+1} with the following procedure:
  1. find the least significant digit i of x^k such that $x_i^k < |\mathcal{A}_i|$;
  2. set $x_i^{k+1} = x_i^k + 1$;
  3. set $x_i^{k+1} = 0$ for all $j < i$;
  4. set $x_i^{k+1} = x_i^k$ for all $j > i$;

  compute the value of $\phi(x^{k+1})$ with the following procedure:
  1. initialize $y = \mathbf{0}$;
  2. set $y_i = 0$;
  3. set $y_j = x_j^{k+1}$ for all $j > i$;
  4. set $y_j = 0$ for all $j < i$;

  set $\phi(x^{k+1}) = \phi(y) + u_i(x^{k+1}) - u_i(y)$;
end
set $\mathcal{L} = \emptyset$;
foreach $\mathcal{K} \subset \mathcal{V}$ **do**
  foreach *$\mathcal{K}$-cube Q* **do**
    if $\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} \phi(x_Q^b) \neq 0$ **then**
      add $\mathcal{K}$ to $\mathcal{L}$;
    end
  end
end
foreach *maximal set $\mathcal{K} \in \mathcal{L}$* **do**
  foreach $i \in \mathcal{K}$ **do**
    set $\mathcal{J} = \mathcal{K} \setminus \{i\}$;
    add the hyperlink $(i, \mathcal{J})$ to $\mathcal{D}$;
  end
end

The first part of Algorithm 2 constructs a potential function for the input potential game. Then, all groups of players are checked for mutual dependency of their members, which can be done directly on the potential function instead of checking each single utility function. This gives the minimal H-graph of the potential function, from which the undirected FDH-graph of the game can be easily obtained exploiting Theorem 7.1.1.

With respect to the generic Algorithm 1, the computational complexity of algorithm 2 for potential games is reduced by a factor $n = |\mathcal{V}|$.

An object oriented implementation of both algorithms has been developed using the Java programming language and is available at the following repository separability.

8.3 SEPARABILITY OF GAMES WITH INCOMPLETE DATA

In the previous sections we have proposed algorithms to find the minimal FDH-graph of a game. They exploit Proposition 8.2.1, which provides checkable conditions to identify players' dependencies in a game based on the geometric characterization of high-order interactions derived in Theorem 8.1.6. We point out that in order to apply Algorithm 1 and 2 full knowledge of the game is assumed: we must know its structure or *form*, which is encoded in the game configuration graph, and all utility values. This is a strong requirement and it is natural to assume that in a realistic setting the game form is know while we do not have access to all the data about utilities of players. This is the case, for example, when we aim to learn the graphical structure of a game by observing the evolution of its outcomes, driven by some game dynamics, or when, for some reason, we have access to just a sample of the utility values.

In this section we will investigate this setting, which is especially interesting from the point of view of applications. We will first describe formally the incomplete data setting and then we will address a fundamental question: what is the best information we can extract from partial data about a game? We will then propose a procedure to learn the hypergraphical structure of a game in such context and we will compare it with the previous Algorithms 1 and 2.

8.3.1 *Games with incomplete data*

Suppose that, instead of a complete specification of a game u, which is a finite set of players $\mathcal{V}$, a finite set of actions for each player $\{\mathcal{A}_i\}_{i\in\mathcal{V}}$ and a family of utility functions $\{u_i\}_{i\in\mathcal{V}}$, we only have partial data about u. To formally define this notion, we first give some definition.

Definition 8.3.1. *A* game form *is a couple* $(\mathcal{V}, \mathcal{X})$ *where* $\mathcal{V}$ *represents a set of players and* $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ *is a configurations space, which is the cartesian product of the action sets* $\mathcal{A}_i$ *of each player.*

Definition 8.3.2. *A game with incomplete data* u^D *is a game form* $(\mathcal{V}, \mathcal{X})$ *and a data set* $D = \{(x_k, u(x_k))\}_{x_k \in \mathcal{X}^D}$*, which specifies utility values* $u(x_k) := \{u_i(x_k)\}_{i \in \mathcal{V}}$ *corresponding to a set of configurations* $\{x_k\}_k = \mathcal{X}^D \subset \mathcal{X}$*.*

In the following we will focus on potential games with incomplete data. The results can be extended to general games but we present them in the potential setting as it allows for an easier formulation. In this case we assume that the data set D specifies the value of a potential function ϕ corresponding to a subset of configurations.

Definition 8.3.3. *A potential game with incomplete data* u^D *is a game form* $(\mathcal{V}, \mathcal{X})$ *and a data set* $D = \{(x_k, \phi(x_k))\}_{x_k \in \mathcal{X}^D}$*, which specifies potential values* $\phi(x_k)$ *corresponding to configurations* $\{x_k\}_k = \mathcal{X}^D \subset \mathcal{X}$*.*

In what follows, an incomplete data game $u^D = (\mathcal{V}, \mathcal{X}, D)$ will represent an underlying (complete) normal form game $u = (\mathcal{V}, \mathcal{X}, \{u_i\}_{i \in \mathcal{V}})$ for which we only have access to information contained in the data set D, relative to configurations $\mathcal{X}^D$. Clearly, if the data set D specifies potential values for the whole configuration space $\mathcal{X}$, then D defines a (complete) normal form game u^D, i.e., D identifies completely the underlying game. Conversely, there are multiple ways to extend the data set D to the whole $\mathcal{X}$, by setting

$$D' = D \cup \{(x_j, \phi(x_j))\}_{x_j \in \mathcal{X} \setminus \mathcal{X}^D}, \tag{117}$$

and each extension leads to a different (complete) potential game $u^{D'}$. Notice that extending the data set D to the whole configuration space is equivalent to extending the potential function $\phi : \mathcal{X}^D \to \mathbb{R}$ to $\mathcal{X}$ and we will use the two notions interchangeably. In the following we will assume a fixed game form $(\mathcal{V}, \mathcal{X})$, and we will focus on the data set D. To stress this fact, we use the following terminology.

Definition 8.3.4. *Games* $u^{D'}$ *associated to an extension* D' *of a data set* D *to the whole* $\mathcal{X}$ *are said to be* compatible *with* D*.*

Any game u that is compatible with D can be thought of as the underlying game that produced the data D through a sampling procedure. Consequently, we give the following definition.

Definition 8.3.5. *An undirected hypergraph* $\mathcal{H}$ *is said to be* compatible *with the data set* D *if there exists a potential game* u *compatible with* D *that is* $\mathcal{F}^{\mathcal{H}}$*-separable.*

We focus on undirected hypergraphs since the separability of potential games is completely determined by the undirected H-graphs of their potential functions, as shown in Theorem 7.1.1.

8.3.2 *Inferring interactions from partial data*

We are now able to address the first questions: what can be said about the separability of a game with incomplete data? To answer this we need to understand how Algorithm 2 behaves when data about the potential function are incomplete.

Algorithm 2 starts with an empty hypergraph, i.e., by assuming no interactions between players, and constructs the minimal hypergraph of a game by repeatedly checking groups of players for mutual interaction, i.e., by checking condition (116) for all cubes associated to each group of players (as by Definition 8.1.1). So, to check (116) for a group $\mathcal{K} \subset \mathcal{V}$ we need potential values corresponding to each vertex of all $\mathcal{K}$-cubes. Even if the data set D is large, it may not have such structure, so that information about some vertex is missing.

A first approach may be to modify Algorithm 2 so that it only checks (116) on a $\mathcal{K}$-cube when the data set D contains information about all its vertices. However, in the worst case, no check can be performed for a group and the modified algorithm would not add any hyperlink to the resulting H-graph to represent the interaction among players in such group. This approach is not correct, as the resulting hypergraph may not be compatible with the data set D, as the following example shows.

Example 8.3.6. *Consider a data set $D = \{(x_k, \phi(x_k))\}_{x_k \in \mathcal{X}^D}$ and a group of two players $\mathcal{K} = \{i, j\}$. Suppose $\mathcal{X}^D = \{x_1, x_2, x_3, x_5, x_6, x_7\}$ does not cover any full cube associated to $\mathcal{K}$. In particular, it does not cover the two cubes $Q_1 = \{x_1, x_2, x_3, x_4\}$ and $Q_2 = \{x_4, x_5, x_6, x_7\}$ because $x_4 \notin \mathcal{X}^D$. However, configurations x_1, x_2, x_3 and x_5, x_6, x_7 are in $\mathcal{X}^D$. Notice that the two cubes Q_1 and Q_2 are both incident on vertex x_4. This setting is represented in Figure 36.*

The check (116) cannot be performed for $\mathcal{K}$ on neither cube, but concluding that players in $\mathcal{K}$ are independent of each other is not correct in general. Indeed, with this reasoning we are neglecting a piece of information, which is related to how different "partial" cubes in the data set interact. To see this, observe that equation (116) is satisfied on Q_1 if

$$\phi(x_1) - \phi(x_2) + \phi(x_3) = \phi(x_4) \tag{118}$$

while it is satisfied on Q_2 if

$$\phi(x_5) - \phi(x_6) + \phi(x_7) = \phi(x_4)\,. \tag{119}$$

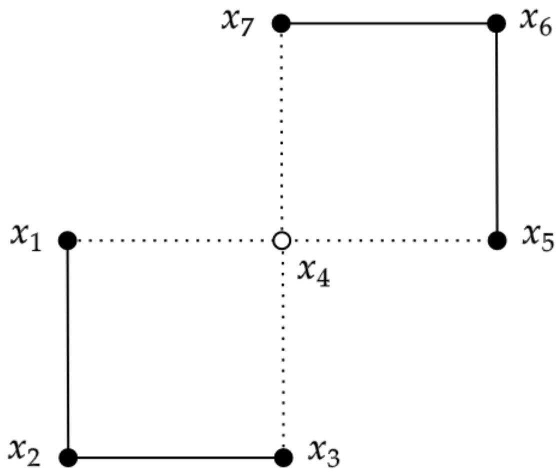


Figure 36: Representation of the setting for Example 8.3.6. Black points correspond to configurations in $\mathcal{X}^D$ while the white point corresponds to configuration x_4, which is not in $\mathcal{X}^D$. The lines highlight two cubes Q_1 and Q_2 relative to the set $\mathcal{K}$: both cubes are not contained in $\mathcal{X}^D$, as they share a missing vertex x_4.

Then, if $\phi(x_1) - \phi(x_2) + \phi(x_3) \neq \phi(x_5) - \phi(x_6) + \phi(x_7)$, we can deduce that in any hypergraph compatible with D, $\mathcal{K}$ is contained in an hyperlink, i.e., players i and j influence each other. Notice that any hypergraph where $\mathcal{K}$ is not contained in some hyperlink is not compatible with D.

Example 8.3.6 shows that there is some information about interactions among players that can be extracted from partial cubes in the data set sharing common missing vertices. In Example 8.3.6, two cubes associated to the same player set $\mathcal{K}$ share a single common missing vertex, but the same reasoning holds more in general whenever partial cubes in the data set share some vertices.

Indeed, while to each cube $Q \subset \mathcal{X}^D$ associated to a group of players $\mathcal{K} \subset \mathcal{V}$, i.e., to each cube such that the data set D contains potential values for all vertices, it corresponds a checkable condition of the form (116), to each cube $Q \not\subset \mathcal{X}^D$, i.e., to each cube such that the data set D is missing the potential value for at least one vertex, it corresponds a linear constraint on the extensions of the data set, i.e., on the values of the potential ϕ outside $\mathcal{X}^D$. Such constraints determine, for each extension of D, if players in $\mathcal{K}$ will form a "dependence group" in the resulting potential game. In other words, the constraints determine, for each extension of D, if the corresponding minimal hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ will contain an hyperlink $\mathcal{J} \in \mathcal{L}$ such that $\mathcal{K} \subset \mathcal{J}$.

In view of this, differently from the case of a complete data game, we cannot define the notion of minimal hypergraph for an incomplete data game. Instead, by joining all constraints (relative to all groups of players $\mathcal{K} \subset \mathcal{V}$) we can define a family of hypergraphs that are compatible with the data set D. Such family does not contain a minimal element and it is not closed with respect to intersection. However, such family will include the minimal hypergraph of the underlying game, from which the data set D was sampled.

8.3.3 *Constructing compatible hypergraphs*

According to the previous observations, for dealing with incomplete data we will define a new procedure, which takes into account both sources of information (the checks for all cubes in $\mathcal{X}^D$, the linear constraints for all cubes not in $\mathcal{X}^D$). While Algorithm 2 gives a single hypergraph as output, the new algorithm will produce the family of hypergraphs compatible with D.

Before presenting the pseudo-code of Algorithm 3 we describe the results at the foundation of it, which formalize the reasoning performed in the previous Section 8.3.2.

Consider a data set $D = \{(x_k, \phi(x_k))\}_{x_k \in \mathcal{X}^D}$ for $\mathcal{X}^D \subset \mathcal{X}$, a set of players $\mathcal{K} \subset \mathcal{V}$ and a $\mathcal{K}$-cube $Q \subset \mathcal{X}$. We denote $Q^D \subset Q$ the subset of vertices of Q that are in $\mathcal{X}^D$ and by $Q^{\overline{D}} \subset Q$ the subset of vertices of Q that are not in $\mathcal{X}^D$, i.e., for which the potential values are unknown. By adapting equation (116) to the potential setting we can write

$$\sum_{b \in \{0,1\}^{\mathcal{K}}} (-1)^{\mathbb{1}'b} \phi(x_Q^b) = 0$$
$$\Leftrightarrow \sum_{x \in Q^{\overline{D}}} a(x)\phi(x) = \sum_{x \in Q^D} -a(x)\phi(x)\,,$$

where $a(x)$ counts the parity of the label of configuration x in the cube Q, i.e., $a(x_Q^b) = (-1)^{\mathbb{1}'b}$. We can represent all linear equations relative to cubes associated to a group $\mathcal{K}$ in matrix form. This is done by first defining a matrix $A^{\mathcal{K}} = (a^{\mathcal{K}}_{Q,x})$ where the row index varies over $\mathcal{K}$-cubes Q and the column index varies over configurations $x \in \mathcal{X}^{\overline{D}} = \mathcal{X} \setminus \mathcal{X}^D$:

$$a^{\mathcal{K}}_{Q,x} = \begin{cases} 0 \text{ if } x \in \mathcal{X}^{\overline{D}} \setminus Q^{\overline{D}} \\ a(x) \text{ if } x \in Q^{\overline{D}}\,. \end{cases} \tag{120}$$

We also define a vector $c^{\mathcal{K}} = (c^{\mathcal{K}}_Q)$, with row index varying over $\mathcal{K}$-cubes Q:

$$c^{\mathcal{K}}_Q := \sum_{x \in Q^D} -a(x)\phi(x)\,. \tag{121}$$

Finally we have the linear system

$$A^{\mathcal{K}}\phi = c^{\mathcal{K}} \tag{122}$$

As stated in the following result, system (122) encodes all information about the player set $\mathcal{K}$ that can be extracted from the data set D.

Lemma 8.3.7. *Consider a data set $D = \{(x_k, \phi(x_k))\}_{x_k \in \mathcal{X}^D}$ with $\mathcal{X}^D \subset \mathcal{X}$, a set of players $\mathcal{K} \subset \mathcal{V}$ and a $\mathcal{K}$-cube $Q \in \mathcal{X}$.*

1. *If the linear system (122) has no solution, then for any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ compatible with D there exists an hyperlink $\mathcal{J} \in \mathcal{L}$ such that $\mathcal{K} \subset \mathcal{J}$.*

2. *If the linear system (122) admits solutions, then the affine space of solutions $\Phi^{\mathcal{K}}$ is the set of potential functions ϕ that extend the data set D to $\mathcal{X}$ for which the minimal hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is such that $\mathcal{K} \not\subset \mathcal{J}$, for all $\mathcal{J} \in \mathcal{L}$.*

In words, the first point of Lemma 8.3.7 states that if the linear constraints for the set $\mathcal{K}$ are incompatible, then players in $\mathcal{K}$ are not independent in any game compatible with the data set D. This means that players in $\mathcal{K}$ are jointly dependent on each other, and they, or a superset of them, form an hyperlink in every hypergraph compatible with D. On the other side, the second point of Lemma 8.3.7 states that if the linear constraints are compatible with each other, then there will be some games compatible with the data set D for which players in $\mathcal{K}$ are not jointly dependent on each other. We are able to characterize such games by looking at the solutions of the linear system (122), and for all such games $\mathcal{K}$ is not contained in any hyperlink of the minimal hypergraph.

Lemma 8.3.7 completely characterizes joint interactions among players in a set $\mathcal{K} \subset \mathcal{V}$ with respect to the information in the data set D. Expanding on this, we can combine the information relative to different groups of players to derive a result characterizing hypergraphs compatible with D.

To show how the information associated to multiple different groups of players can be combined, we first present the following example where only two groups are present.

Consider two sets of players $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{V}$, for which the linear system (122) admits solutions and denote the spaces of solutions by $\Phi^{\mathcal{K}_1}$ and $\Phi^{\mathcal{K}_2}$ respectively. If $\Phi^{\mathcal{K}_1} \cap \Phi^{\mathcal{K}_2} = \emptyset$, it means that in all games compatible with the data set D at least one of the two groups of players are jointly dependent on each other, since there is no way to satisfy the constraints for $\mathcal{K}_1$ and $\mathcal{K}_2$ simultaneously. As a consequence, for any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ compatible with D there exists an hyperlink $\mathcal{J} \in \mathcal{L}$ such that $\mathcal{K}_1 \subset \mathcal{J}$ or $\mathcal{K}_2 \subset \mathcal{J}$. If, instead, $\Phi^{\mathcal{K}_1} \cap \Phi^{\mathcal{K}_2} \neq \emptyset$, it is possible to satisfy the constraints for $\mathcal{K}_1$ and $\mathcal{K}_2$ simultaneously. Then, there exists at least one game compatible with D for which both groups of players are not jointly dependent. As a consequence, there exists at least one potential function ϕ that extends the data D to $\mathcal{X}$ for which the minimal hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is such that for all $\mathcal{J} \in \mathcal{L}$, $\mathcal{K}_i \not\subset \mathcal{J}$, for all $i \in \{1, 2\}$.

More in general, we can state the following.

Theorem 8.3.8. *Consider a family G of k subsets of the player set $\mathcal{V}$, $G \subset \mathcal{P}(\mathcal{V})$, and assume that for each subset $\mathcal{K} \in G$ the linear system (122) has a non-empty solution space $\Phi^{\mathcal{K}} \neq \emptyset$.*

1. *if $\bigcap_{\mathcal{K} \in G} \Phi^{\mathcal{K}} = \emptyset$ then for any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ compatible with D there exists an hyperlink $\mathcal{J} \in \mathcal{L}$ such that $\mathcal{K} \subset \mathcal{J}$ for some $\mathcal{K} \in G$;*
2. *if $\bigcap_{\mathcal{K} \in G} \Phi^{\mathcal{K}} \neq \emptyset$, then there exists at least one potential function ϕ that extends the data set D to $\mathcal{X}$ for which the minimal hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ is such that for all $\mathcal{K} \in G$ and for all $\mathcal{J} \in \mathcal{L}$, $\mathcal{K} \not\subset \mathcal{J}$.*

In view of this result, we are able to characterize the family of all hypergraphs compatible with a given data set D.

Corollary 8.3.9. *Let F be a collection of families of player sets such that for all $G \in F$, $\bigcap_{\mathcal{K} \in G} \Phi^{\mathcal{K}} = \emptyset$. Then for any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ compatible with D and for each G there exists an hyperlink $\mathcal{J} \in \mathcal{L}$ such that $\mathcal{K} \subset \mathcal{J}$ for some $\mathcal{K} \in G$.*

When F collects all families of player sets $G \subset \mathcal{P}(\mathcal{V})$ for which $\bigcap_{\mathcal{K} \in G} \Phi^{\mathcal{K}_i} = \emptyset$, Corollary 8.3.9 gives a characterization that consists of an exhaustive set of constraints. It can be expressed as a conjunction of logical disjunctions of statements of the kind $p(\mathcal{K})$ = "players in the set $\mathcal{K} \subset \mathcal{V}$ are jointly dependent", as follows

$$\wedge_{G \in F} \left(\vee_{\mathcal{K} \in G} \, p(\mathcal{K}) \right) \tag{123}$$

Indeed, by the first point of Theorem 8.3.8, players in at least one set for each family are jointly dependent. With only the partial information contained in the data set D, we cannot know exactly which of them is and this gives rise to a family of compatible hypergraphs: each hypergraph corresponds to an assignment of logical values to statements $p(\mathcal{K})$ which makes (123) true.

All previous results can be summarized in the following algorithmic procedure for constructing the characterization F of the family of hypergraphs $\mathcal{H}$ compatible with a data set D.

Algorithmus 3 : Compute the family of H-graphs for a potential game u^D with incomplete data.

Result : characterization of the family of H-graphs $\mathcal{H}$ compatible with D
Input : game-form $(\mathcal{V}, \mathcal{X})$, data set $D = \{(x_k, \phi(x_k))\}_{x_k \in \mathcal{X}^D}$ with $\mathcal{X}^D \subset \mathcal{X}$
Initialization: set $F = \emptyset$;
foreach *family of sets* $G \subset \mathcal{P}(\mathcal{V})$ **do**
    foreach $\mathcal{K} \in G$ **do**
        compute the solutions $\Phi^{\mathcal{K}}$ of the linear system $A^{\mathcal{K}}\phi = c^{\mathcal{K}}$;
    end
    if $\bigcap_{\mathcal{K} \in G} \Phi^{\mathcal{K}} = \emptyset$ **then**
        add G to F
    end
end

Note that the collection F constructed with Algorithm 3 may contain redundant information and can in general be refined. For example, if F contains two families $G_1 = \{A, B\}$ and $G_2 = \{A, B, C\}$, then we can delete G_2 from F and only keep G_1. Indeed, since $G_1 \subset G_2$, G_1 corresponds to a more specific statement: at least one between A and B is a set of jointly dependent players. However, this kind of considerations are not substantial and more pertaining to the implementation level.

9

CONCLUSIONS

9.1 CONCLUSION

In this thesis we have developed a systematic study of separable games, a game theoretic representation that encompasses several models receiving great attention in both the theoretic and applied game theory literature and that generalizes both polymatrix and graphical games first introduced in the foundational works [86, 59]. Specifically, the separable game representation introduced in Chapter 3 is able to precisely describe the structure of strategic interactions in games by explicitly representing arbitrary complex patterns of dependence among players by means of forward directed hypergraphs. We have analysed this representation in details in Chapter 4, where we have showed how different separable representations for a single game can be combined and we have proved the existence of a minimal separable representation for any game. We have discussed complexity issues related to such representation and we have proposed the notion of strict separability that allows to represent games compactly, as it may be desirable in computational settings.

Throughout the thesis we have mainly undertaken a modelling perspective, focusing on the expressive power of the separable representation of games and on how it can be leveraged in modelling and analysis of games. In this spirit we have devoted Chapters 6 and 7 to describe the implications of separability on the properties of games.

First, in Chapter 6 we have considered generic separable games and we have shown that the structure of their correlated equilibria is shaped by the FDH-graph expressing their separability. Precisely, we have proven that correlated equilibria of an $\mathcal{F}$-separable game can be factorized over hyperlinks of $\mathcal{H}^{\mathcal{F}}$, up to $\mathcal{H}^{\mathcal{F}}$-hyperlink equivalence. Moreover, we have identified a geometric assumption on $\mathcal{H}^{\mathcal{F}}$, namely decomposability, which allows to efficiently construct factorizable correlated equilibria with prescribed local hyperlink marginals or that are optimal according to a class of linear objectives.

Then, in Chapter 7 we have focused on potential games, a class that plays a major role in game theory. We have characterized the structure of separable potential games, showing that their minimal FDH-graph is undirected to reflect a symmetry of interactions among players. We have leveraged concepts from the theory of Markov random fields to highlight the connection between separability of potential games and Markov properties of the probabilistic graphical models associated to their potential function. Within this framework, we have also given new insights on the Hammersley-Clifford theorem for graphical models, proposing a novel game theoretic derivation of it. Finally, we have exploited these findings to generalize or improve some results present in the literature, specifically to characterize the interplay of separability with the potential-harmonic decomposition of [31] and to obtain bounds on the length of better response paths in separable potential games on the lines of [14].

Even if they hold in general, the structural results we have obtained are most interesting when games exhibit a "fine", non-trivial separability. This happens frequently when separability of games is a design or an emergent property, but it may not always be the case. For this reason, before moving to describing the way separability reflects on relevant game properties, in Chapter 5 we have first shown how to obtain separable approximations of games with a prescribed hypergraphical structure by means of projections. By carefully balancing between goodness of the approximation and tractability of the hypergraphical structure, this approach can be useful when analysing complex games with many players and a dense pattern of interactions.

Finally, in Chapter 8 we have assumed a computational perspective and we have proposed algorithms to check minimality of given separable representations or to obtain the minimal separable representation of games. These are based on a geometric characterization of separability that we have derived, which expresses separability of a game in terms of linear constraints on its utility values associated to cubic subgraphs of the game configuration graph. The analysis we have developed offers a way to practically compute the minimal FDH-graph of given games, also covering the case when only partial data about the game are available by introducing the notion of FDH-graph compatible with some data, and, most importantly, it provides an operative definition of minimal separability to build on.

In conclusion, in this thesis we have introduced the separable game model and we have developed a broad analysis of it, starting from the more fundamental and theoretic aspects and comprising computational aspects and applications. The result is a comprehensive treatment of this novel, still largely unexplored game representation.

9.2 FUTURE DEVELOPMENTS

The product of this thesis can be the starting point for many lines of future research, of which here we briefly present some of the most promising.

The problem of learning graphical games is attracting an ever growing attention in the literature. Indeed, the literature on graphical games has mainly been centered on analysing games properties given their graph structure and to investigate the effects of such known structure on their emerging characteristics [52, 27, 16, 44]. However, in many real world settings, access to the network structure of a game is difficult if not impossible. This is the case, for example, in social settings where interactions among players are not a design element but the result of some network formation process driven by social and personal preferences of individuals, which are hard to capture, observe and model. This is also true when the network of interactions is conceptually accessible but it is too large or complex to be fully explored, as it is the case for many on-line networks, such as the Internet. In all these cases, particularly interesting is the problem of learning the structure of games. This is a sort of inverse problem, where one tries to infer the network of interactions from observations of players behaviour. Some approaches assume players payoffs to be accessible [41] while the game evolves dynamically and focus only on structure learning, while others relax this assumption by only relying on noisy observations of payoffs [17]. Either way, these assumption are strong, since while players actions are often public, their resulting payoff is usually a private information. Research is then focusing on learning graphical games by only observing players actions. As pointed out in [61], learning utilities alone, under the assumption that they can be represented by a small number of parameters, is itself a challenging problem but jointly inferring the utilities and graph structure of a game is very hard. Various approaches have been proposed for this task, which restrict learning to some classes of parametrized games and are grounded on the solution concept of pure strategy Nash equilibria. On this line we cite two of the main contributions, focused on learning linear influence games [49] and linear quadratic games on networks [65]. To the best of our knowledge, all the available literature focuses on assessing interaction between players at the binary level. We think that a very interesting evolution of this line of research would regard learning the structure of separable games, i.e., inferring joint dependences among players. Hopefully, this thesis and the understanding it provides on the separable representation of games can be a foundation and a starting point for the development of new approaches.

Another interesting line of research would concern learning *in* separable games. This is a different problem in which the focus is on how players learn to play a game by repeatedly interacting with each other as the game evolves dynamically and on how they can achieve stable outcomes, such as Nash equilibria. There is a large literature of learning in games [43] and, specifically, on learning in graphical games [1, 2, 83]. These works take advantage of the graphical structure of games to design learning schemas and algorithms and we argue that similar approaches could successfully leverage separability of games.

BIBLIOGRAPHY

[1] Mohammed I Abouheaf, Frank L Lewis, Kyriakos G Vamvoudakis, Sofie Haesaert, and Robert Babuska. Multi-agent discrete-time graphical games and reinforcement learning solutions. *Automatica*, 50(12):3038–3053, 2014.

[2] Mohammed I Abouheaf, Frank L Lewis, Magdi S Mahmoud, and Dariusz G Mikulski. Discrete-time dynamic graphical games: model-free reinforcement learning solution. *Control Theory and Technology*, 13(1):55–69, 2015.

[3] Peter A Abrams. Arguments in favor of higher order interactions. *The American Naturalist*, 121(6):887–891, 1983.

[4] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Networks, shocks, and systemic risk. Technical report, National Bureau of Economic Research, 2015.

[5] Nizar Allouch. On the private provision of public goods on networks. *Journal of Economic Theory*, 157:527–552, 2015.

[6] Alessandro Aloisio, Michele Flammini, and Cosimo Vinci. The impact of selfishness in hypergraph hedonic games. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 1766–1773, 2020.

[7] Unai Alvarez-Rodriguez, Federico Battiston, Guilherme Ferraz de Arruda, Yamir Moreno, Matjaž Perc, and Vito Latora. Evolutionary dynamics of higher-order interactions in social networks. *Nature Human Behaviour*, 5(5): 586–595, 2021.

[8] Laura Arditti, Giacomo Como, and Fabio Fagnani. Graphical games and decomposition. *arXiv preprint arXiv:2003.13123*, 2020.

[9] Laura Arditti, Giacomo Como, and Fabio Fagnani. Separable games. *arXiv preprint arXiv:2003.13128*, 2020.

[10] Laura Arditti, Giacomo Como, Fabio Fagnani, and Martina Vanelli. Equilibria and learning dynamics in mixed network coordination/anti-coordination games. *arXiv preprint arXiv:2109.12692*, 2021.

[11] Laura Arditti, Giacomo Como, Fabio Fagnani, and Martina Vanelli. Robust coordination in networks. *arXiv preprint arXiv:2109.12685*, 2021.

[12] Robert J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96, 1974. ISSN 0304-4068. doi: https://doi.org/10.1016/0304-4068(74)90037-8. URL https://www.sciencedirect.com/science/article/pii/0304406874900378.

[13] Robert J Aumann. Correlated equilibrium as an expression of bayesian rationality. *Econometrica: Journal of the Econometric Society*, pages 1–18, 1987.

[14] Y. Babichenko and O. Tamuz. Graphical potential games. *Journal of Economic Theory*, 163:889–899, 2016.

[15] Eyal Bairey, Eric D Kelsic, and Roy Kishony. High-order species interactions shape ecosystem diversity. *Nature communications*, 7(1):1–7, 2016.

[16] C. Ballester, A. Calvó-Armengol, and Y. Zenou. Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417, 2006.

[17] Adarsh Barik and Jean Honorio. Provable sample complexity guarantees for learning of continuous-action graphical games with nonparametric utilities. *arXiv preprint arXiv:2004.01022*, 2020.

[18] Federico Battiston, Giulia Cencetti, Iacopo Iacopini, Vito Latora, Maxime Lucas, Alice Patania, Jean-Gabriel Young, and Giovanni Petri. Networks beyond pairwise interactions: structure and dynamics. *Physics Reports*, 874:1–92, 2020.

[19] Federico Battiston, Enrico Amico, Alain Barrat, Ginestra Bianconi, Guilherme Ferraz de Arruda, Benedetta Franceschiello, Iacopo Iacopini, Sonia Kéfi, Vito Latora, Yamir Moreno, et al. The physics of higher-order interactions in complex systems. *Nature Physics*, 17(10):1093–1098, 2021.

[20] Austin R Benson, David F Gleich, and Desmond J Higham. Higher-order network analysis takes off, fueled by classical ideas and new data. *arXiv preprint arXiv:2103.05031*, 2021.

[21] Adam Berger, Stephen A Della Pietra, and Vincent J Della Pietra. A maximum entropy approach to natural language processing. *Computational linguistics*, 22(1):39–71, 1996.

[22] Christian Bick, Peter Ashwin, and Ana Rodrigues. Chaos in generically coupled phase oscillator networks with nonpairwise interactions. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 26(9):094814, 2016.

[23] Ágnes Bodó, Gyula Y Katona, and Péter L Simon. Sis epidemic propagation on hypergraphs. *Bulletin of mathematical biology*, 78(4):713–735, 2016.

[24] A. Bondy and U.S.R. Murty. *Graph Theory*. Graduate Texts in Mathematics. ISBN 9781846289699.

[25] Yann Bramoullé. Anti-coordination and social interactions. *Games and Economic Behavior*, 58(1):30–49, 2007.

[26] Yann Bramoullé and Rachel Kranton. Public goods in networks. *Journal of Economic theory*, 135(1):478–494, 2007.

[27] Yann Bramoullé and Rachel Kranton. Games played on networks. 2015.

[28] A. Bretto. *Hypergraph Theory: An Introduction*. Springer, 2013.

[29] Y. Cai and C. Daskalakis. On minmax theorems for multiplayer games. In *Proceeding Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete algorithms (SODA '11)*, pages 217–234, 2011.

[30] Yang Cai, Ozan Candogan, Constantinos Daskalakis, and Christos Papadimitriou. Zero-sum polymatrix games: A generalization of minmax. *Mathematics of Operations Research*, 41(2):648–655, 2016.

[31] O. Candogan, I. Menache, A. Ozdaglar, and P.A. Parrilo. Flows and decompositions of games: Harmonic and potential games. *Mathematics of Operations Research*, 36(3):474–503, 2011.

[32] Timoteo Carletti, Duccio Fanelli, and Sara Nicoletti. Dynamical systems on hypergraphs. *Journal of Physics: Complexity*, 1(3):035006, 2020.

[33] Fan Chung and Alexander Tsiatas. Hypergraph coloring games and voter models. In *International Workshop on Algorithms and Models for the Web-Graph*, pages 1–16. Springer, 2012.

[34] C. Daskalakis and C.H. Papadimitriou. On a network generalization of the minmax theorem. In *Proceedings of the 36th Internatilonal Collogquium on Automata, Languages and Programming (ICALP '09): Part II*, pages 423–434, 2009.

[35] Constantinos Daskalakis and Christos H Papadimitriou. Three-player games are hard. In *Electronic colloquium on computational complexity*, volume 139, pages 81–87. Citeseer, 2005.

[36] Constantinos Daskalakis and Christos H Papadimitriou. Computing pure nash equilibria in graphical games via markov random fields. In *Proceedings of the 7th ACM Conference on Electronic Commerce*, pages 91–99, 2006.

[37] Constantinos Daskalakis and Christos H Papadimitriou. On a network generalization of the minmax theorem. In *International Colloquium on Automata, Languages, and Programming*, pages 423–434. Springer, 2009.

[38] Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou. The complexity of computing a nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259, 2009.

[39] A Philip Dawid and Steffen L Lauritzen. Hyper markov laws in the statistical analysis of decomposable graphical models. *The Annals of Statistics*, pages 1272–1317, 1993.

[40] Guilherme Ferraz de Arruda, Giovanni Petri, and Yamir Moreno. Social contagion models on hypergraphs. *Physical Review Research*, 2(2):023032, 2020.

[41] Quang Duong, Yevgeniy Vorobeychik, Satinder Singh, and Michael Wellman. Learning graphical game models. In *Twenty-First International Joint Conference on Artificial Intelligence*, 2009.

[42] David Easley and Jon Kleinberg. Networks, crowds, and markets. reasoning abouth a highly connected world. *Cambridge University Press*, 2010.

[43] Drew Fudenberg, Fudenberg Drew, David K Levine, and David K Levine. *The theory of learning in games*, volume 2. MIT press, 1998.

[44] Andrea Galeotti, Benjamin Golub, and Sanjeev Goyal. Targeting interventions in networks. *Econometrica*, 88(6):2445–2471, 2020.

[45] Giorgio Gallo, Giustino Longo, Stefano Pallottino, and Sang Nguyen. Directed hypergraphs and applications. *Discrete Applied Mathematics*, 42(2):177 – 201, 1993. ISSN 0166-218X. doi: https://doi.org/10.1016/0166-218X(93)90045-P. URL http://www.sciencedirect.com/science/article/pii/0166218X9390045P.

[46] Elad Ganmor, Ronen Segev, and Elad Schneidman. Sparse low-order interaction network underlies a highly correlated and learnable neural population code. *Proceedings of the National Academy of sciences*, 108(23):9679–9684, 2011.

[47] Georg Gottlob, Gianluigi Greco, and Francesco Scarcello. Pure nash equilibria: Hard and easy games. *Journal of Artificial Intelligence Research*, 24:357–406, 2005.

[48] Jacopo Grilli, György Barabás, Matthew J Michalska-Smith, and Stefano Allesina. Higher-order interactions stabilize dynamics in competitive network models. *Nature*, 548(7666):210–213, 2017.

[49] Jean Honorio and Luis E Ortiz. Learning the structure and parameters of large-population graphical games from behavioral data. *J. Mach. Learn. Res.*, 16(1):1157–1210, 2015.

[50] Joseph T Howson Jr. Equilibria of polymatrix games. *Management Science*, 18 (5-part-1):312–318, 1972.

[51] Iacopo Iacopini, Giovanni Petri, Alain Barrat, and Vito Latora. Simplicial models of social contagion. *Nature communications*, 10(1):1–9, 2019.

[52] M. O. Jackson and Y. Zenou. *Handbook of game theory with economic applications*, volume 4, chapter Games on networks, pages 95–163. Elsevier, 2015.

[53] Matthew O Jackson and Evan Storms. Behavioral communities and the atomic structure of networks. *Available at SSRN 3049748*, 2019.

[54] M.O. Jackson. *Social and Economic Networks*. Princeton University Press, 2008.

[55] Elena Janovskaja. Equilibrium points in polymatrix games. *Lithuanian Mathematical Journal*, 8(2):381–384, 1968.

[56] Edwin T Jaynes. Prior probabilities. *IEEE Transactions on systems science and cybernetics*, 4(3):227–241, 1968.

[57] Albert Xin Jiang, Kevin Leyton-Brown, and Navin AR Bhat. Action-graph games. *Games and Economic Behavior*, 71(1):141–173, 2011.

[58] Sham Kakade, Michael Kearns, John Langford, and Luis Ortiz. Correlated equilibria in graphical games. In *Proceedings of the 4th ACM Conference on Electronic Commerce*, pages 42–47, 2003.

[59] M. Kearns, M. L. Littman, and S. Singh. Graphical models for game theory. In *Proceedings of the Seventeenth Conference on Uncertainty in Artificial Intelligence (UAI2001)*, pages 253–260, 2001.

[60] Daphne Koller and Brian Milch. Multi-agent influence diagrams for representing and solving games. *Games and economic behavior*, 45(1):181–221, 2003.

[61] Volodymyr Kuleshov and Okke Schrijvers. Inverse game theory: Learning utilities in succinct games. In *International Conference on Web and Internet Economics*, pages 413–427. Springer, 2015.

[62] Pierfrancesco La Mura. Game networks. *arXiv preprint arXiv:1301.3870*, 2013.

[63] Renaud Lambiotte, Martin Rosvall, and Ingo Scholtes. From networks to optimal higher-order models of complex systems. *Nature physics*, 15(4):313–320, 2019.

[64] S. L. Lauritzen. *Graphical models*. Oxford Statistical Science Series. Oxford University Press, 1996.

[65] Yan Leng, Xiaowen Dong, Junfeng Wu, and Alex Pentland. Learning quadratic games on networks. In *International Conference on Machine Learning*, pages 5820–5830. PMLR, 2020.

[66] Jonathan M Levine, Jordi Bascompte, Peter B Adler, and Stefano Allesina. Beyond pairwise mechanisms of species coexistence in complex communities. *Nature*, 546(7656):56–64, 2017.

[67] Margaret M Mayfield and Daniel B Stouffer. Higher-order interactions capture unexplained complexity in diverse communities. *Nature ecology & evolution*, 1(3):1–7, 2017.

[68] Nimrod Megiddo and Christos H Papadimitriou. On total functions, existence theorems and computational complexity. *Theoretical Computer Science*, 81(2):317–324, 1991.

[69] D. Monderer and L. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[70] Andrea Montanari and Amin Saberi. The spread of innovations in social networks. *Proceedings of the National Academy of Sciences*, 107(47):20196–20201, 2010.

[71] Stephen Morris. Contagion. *The Review of Economic Studies*, 67(1):57–78, 2000.

[72] John Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 1950.

[73] John Nash. Non-cooperative games. *Annals of mathematics*, pages 286–295, 1951.

[74] John F Nash Jr. Equilibrium points in n-person games. *Proceedings of the national academy of sciences*, 36(1):48–49, 1950.

[75] J v Neumann. Zur theorie der gesellschaftsspiele. *Mathematische annalen*, 100 (1):295–320, 1928.

[76] Luis E Ortiz. Graphical potential games. *arXiv preprint arXiv:1505.01539*, 2015.

[77] Luis E. Ortiz and M. T. Irfan. Fptas for mixed-strategy nash equilibria in tree graphical games and their generalizations. In *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence (AAAI-17)*, 2017.

[78] Luis E Ortiz, Robert E Schapire, and Sham M Kakade. Maximum entropy correlated equilibria. In *Artificial Intelligence and Statistics*, pages 347–354. PMLR, 2007.

[79] Christos H Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and system Sciences*, 48(3):498–532, 1994.

[80] Christos H Papadimitriou and Tim Roughgarden. Computing correlated equilibria in multi-player games. *Journal of the ACM (JACM)*, 55(3):1–29, 2008.

[81] Sunil Simon and Dominik Wojtczak. Synchronisation games on hypergraphs. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence*. International Joint Conferences on Artificial Intelligence Organization, 2017.

[82] Rann Smorodinsky and Shakhar Smorodinsky. Hypergraphical clustering games of mis-coordination. *arXiv preprint arXiv:1706.05297*, 2017.

[83] Farzaneh Tatari, Mohammad-Bagher Naghibi-Sistani, and Kyriakos G Vamvoudakis. Distributed learning algorithm for non-linear differential graphical games. *Transactions of the Institute of Measurement and Control*, 39 (2):173–182, 2017.

[84] Martina Vanelli, Laura Arditti, Giacomo Como, and Fabio Fagnani. On games with coordinating and anti-coordinating agents. *IFAC-PapersOnLine*, 53(2): 10975–10980, 2020.

[85] Mohamed Wahbi and Kenneth N Brown. A distributed asynchronous solver for nash equilibria in hypergraphical games. In *Frontiers in Artificial Intelligence and Applications*, volume 285, pages 1291–1299. IOS Press, 2016.

[86] Elena B Yanovskaya. Equilibrium points in polymatrix games. *Litovskii Matematicheskii Sbornik*, 8:381–384, 1968.

[87] H Peyton Young. The diffusion of innovations in social networks. *The economy as an evolving complex system III: Current perspectives and future directions*, 267: 39, 2006.